Wits School of Education & Gauteng Department of Education

Maths 4 Teachers

Volume 2 Issue 3a July 2009

In our third issue of the 2009 newsletter we have chosen to talk about how learners work with problems based on space, shape and measurement. We will focus on solving problems in which learners have to work with what they know about areas and perimeters of basic shapes to calculate areas and perimeters of the complex shapes.

INSIDE THIS ISSUE

WHY IS AN ABILITY TO SOLVE SPATIAL PROBLEMS IMPORTANT?

Abilities to solve problems and work spatially are closely related. Problem solving is a valuable skill for many aspects of daily life – not necessarily only at school. Problem-solving at any conceptual level may be understood as an ability to formulate a mathematical problem, represent the problem and then solve it[1]. To formulate a problem people need to first understand what the situation is; after which they need to be able to focus on the "core mathematical elements" of the problem and discard unnecessary information. They then need to be able to use appropriate procedures to solve the problem.

Spatial reasoning in the NCS is addressed in LO3. Almost all of the Assessment Standards in LO3 in all grades are related to each other conceptually. Spatial reasoning is also used to solve many problems found in LO4, and therefore, on many occasions competence in LO4 requires learners to also be competent in LO3.

Some grade 3 DIPIP teachers: Lucinda, Ronica, Anne and Bgwilizani

[1] Kilpatrick, J.; Swafford, J. & Findell, B. (2001). Adding it up: Helping children learn mathematics. Washington DC: National Academy Press

Spatial problems (LO3) are difficult to work with, because they involve aspects of spatial orientation and mental visualisation, with which many learners struggle. Combined with measurement (LO4) they become a lot more challenging because learners need to also strategically use measurement formulae in spatial problems. In this newsletter we combine problem-solving with aspects of space and shape (LO3) and measurement (LO4).

We analyse and discuss two grade 5 and two grade 9 ICAS items which involve problem solving concerning space, shape and measurement. All four items deal with calculating the area of complex shapes. We have written this newsletter in two parts – 3a and 3b. Part 3a works with the grade 5 items and 3b with the grade 9 items. The problems learners face when working with space, shape and measurement in grade 5 are similar to those found in other primary and high school grades. So, what is said in this newsletter applies equally well to all primary and secondary school teachers. For this reason, it is important that you read both issues thoroughly, even though issue 3a focuses on primary school and issue 3b focuses on secondary school learning.

There is no grade 5 assessment standard (AS) that specifies the use of formulae to calculate perimeters and areas of shapes in different orientations or complex shapes. Two of the most relevant ASs we are working with from LOs 3 and 4 (grade 5) are:

From LO3: "… Describes, sorts and compares 2-D shapes and 3-D objects from the environment and from drawings or pictures according to properties including:

- number and/or lengths of sides
- number and/or shape of faces"

From LO4: "… investigates and approximates (alone and/or as a member of a group or team):

- Perimeter using rulers or measuring tapes
- Area of polygons (using square grids and tiling) in order to develop an understanding of square units …"

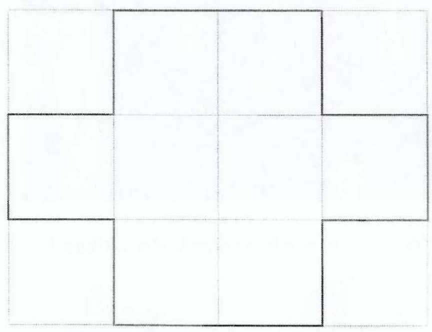
As usual, we discuss the selected items and provide a brief summary of learner performance on each. After each item we provide an analysis of how the correct answer may be obtained. We then discuss the common errors made in each item, and some of the reasons for them.

What the learners did

We discuss two ICAS items below.

Grade 5 item 9:

Holly drew this shape on 2 cm grid paper.



What is the area of Holly's shape?

(A) 32 cm^2
(B) 28 cm^2
(C) 16 cm^2
(D) 8 cm^2

14% of learners arrived at the correct answer (A)
34% chose option D
23% chose option C
22% chose option B

To do this question the learners have to work with "2cm grid paper". The DIPIP teachers who analysed this item said that they tend work only with unit squares in these grades, although the use of unit squares exclusively is not specified in the NCS. Because of the use of non-unit squares, according to them, this question was more suited to grade 7 than grade 5 learners. We point out that whatever is specified in the NCS is the **minimum** that should be taught in any particular grade, and that learners in grade 5 and 6 should be exposed to squares other than 1cm by 1cm in dimensions. The DIPIP teachers acknowledged this after they had analysed the item more thoroughly.

There are a few ways in which the problem could be solved correctly:

1. The area of each little square may be calculated as 4cm². There are 8 little squares inside Holly's shape, so the area of 4cm² multiplied by the number of squares in the shape (8) will result in a total area of 32cm².
2. Some learners could have calculated the area of the whole 8 by 6cm rectangle as 48cm², and then subtracted the area of the four corner squares from 48cm²:
   $48cm^2 - (4 \times 4cm^2) = 32cm^2$.
3. It is also possible to view the top two squares as a rectangle, the next four squares as another rectangle and the bottom two squares as another rectangle. They would then calculate the area of each of the three rectangles and add them:
   $(2cm \times 4cm) + (2cm \times 8cm) + (2cm \times 4cm)$
   $= 8 + 16 + 8\ cm^2$
   $= 32cm^2$

Correct solutions 2 and 3 require learners to know how to work with the formula of a rectangle, which is unusual in grade 5.

In all of the correct solutions learners needed to be able to work with a 2cm by 2cm grid, with which many are not familiar.

Discussion of learner errors

The distractors indicated different aspects of learners' misunderstandings of the complex shape. Approximately a third of the learners chose incorrect answer D (8cm²). To get this answer, they probably counted a total of 8 blocks, but possibly did not understand the information given to them about the dimensions of each block being 2cm by 2cm. Learners may not have understood what "2cm grid paper" was. Since grade 4, 5 and 6 South African learners usually only work with unit squares, the learners who chose this answer most likely assumed that the squares were 1cm by 1cm. Therefore, it was quite logical to count a total of 8 blocks and call the area 8cm².

Nearly a quarter of the learners chose incorrect answer C (16cm²). It is likely that they understood that 2cm × 2cm somehow made the "2cm grid paper" spoken of in the question. Hence, they multiplied 8 squares by 2 to get 16, which was given as a possible answer.

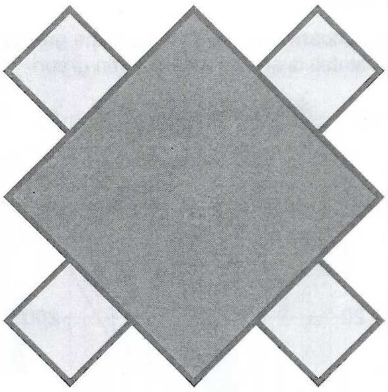
Choosing distractors C or D probably indicates that because the learners have only recently begun to work with the concept of area, they did not understand that when the side of a square doubles the area increases four times.

Almost a quarter of the learners chose incorrect answer B (28cm²). Since the concepts of area and perimeter are very new to grade 5 learners it is possible that they mixed up the concepts and found the perimeter of the shape by adding up all the outside sides of 2cm.

Manono and Sannah (gr 3) presenting their ideas to the larger group of DIPIP teachers

Grade 5 item 29:

This shape has four small squares and one large square.



The area of each small square is 4 cm^2.

What is the area of the **whole** shape?

(A) 13 cm^2
(B) 36 cm^2
(C) 52 cm^2
(D) 160 cm^2

22% of learners arrived at the correct answer (C)
25% chose option B
24% chose option D
21% chose option A

In item 29 the areas of the small squares are 4cm^2, and like the previous item, squares that are not unit squares are likely to be unfamiliar to most grade 5 learners.

There are a few ways in which the problem could be solved correctly:

1. The big square is made up of 9 small squares, each of which is 4cm^2. The area of the big square is therefore 9 x 4 = 36cm^2 .There are 4 small squares, each with an area of 4cm^2, so 4 x 4 = 16cm^2. Add the areas: 16 + 36 = 52cm^2
2. There is a total of 13 small squares in the whole shape. Multiply 13 by 4cm^2, which gives 52cm^2.
3. A more unusual method could be to work with unit lengths of each square. The small squares are therefore 2 x 2 = 4cm^2, and the big square 6 x 6 = 36cm^2. The total is 36 + (4 $\times$ 4) = 52cm^2.

Discussion of learner errors

25% of the learners chose incorrect answer B (36cm^2). They could have said that the big square was made of 9 small squares, each having an area of 4cm^2. They then said 9 x 4 = 36cm^2, but misinterpreted the word "whole" as the big square and not the complete shape; especially because the big square was a different colour from the four smaller squares. It is also possible that in the complexity of the problem they forgot to include the areas of the four small squares in their final answer.

For choice D (160cm^2) learners could have divided the big square into 9 small squares and calculated the area of the big square which is 36. They then added the 4 small squares without considering the areas of the small squares, to 36 which makes 40. They then worked out the area of the whole shape by multiplying 40 x 4 = 160cm^2. This shows a lack of understanding of the concepts of area, as well as a lack of systematic problem-solving skills.

Choosing option A indicates that the learners did not know how to work with squares of area $4cm^2$, or did not keep foremost in their minds that they were working with $4cm^2$ squares. Instead they worked with unit squares. To get an answer of $13cm^2$ they probably divided the big square into 9 small squares, and then they totalled the small squares, to 13 squares at $1cm^2$ each.

A group of DIPIP teachers at the DIPIP 2009 orientation meeting

A CLOSER LOOK AT LEARNERS' PROBLEMS CONCERNING SPACE AND MEASUREMENT

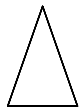
We identified four problems associated with the grade 5 items that can be generalised across primary school grades. They are:

(1) difficulty with shapes which are oriented in an unusual way,
(2) breaking down complex shapes into their basic components
(3) working with grids that are not made up of unit squares, and

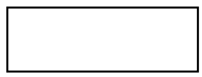
We discuss each, and provide suggestions how to teach to deal with these problems.

(1) Difficulty with shapes which are oriented in an unusual way

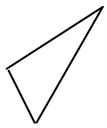
To deal with learners' difficulties in working with shapes in different orientations teachers can fairly easily change the way they work with shapes, from as early as grade 1. Teachers can deliberately present shapes in different orientations to their learners, whenever they draw them on the board or on worksheets. For example, triangles and quadrilaterals should not always be drawn with a side placed horizontally. In addition to drawing:



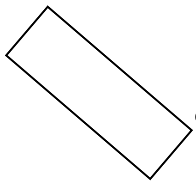
and



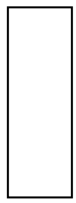
also remember to draw



and



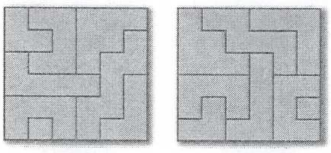
or



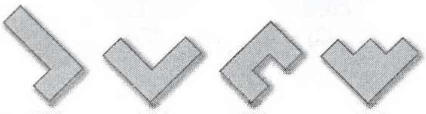
Any shapes and three-dimensional objects can be presented in different orientations when they are first taught. Then learners are less likely to develop the understanding that shapes and objects are associated with any particular orientation, but rather with the properties that make them distinct from each other.

In the higher grades teachers can continue to give problems to their learners that involve shapes in different orientations. For example, grade 5 learners could be given the following questions from the ICAS test, or similar problems, where shape orientations are unusual or altered.

1.
Jenny has two jigsaw puzzles.

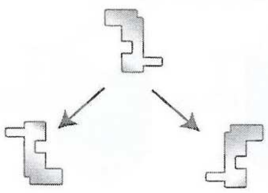


Which one of these pieces can be found in one puzzle but **NOT** in the other?



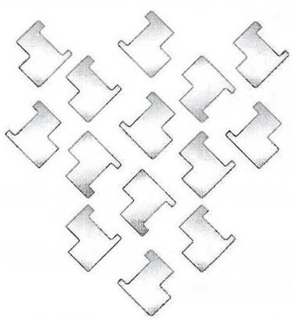
(A) (B) (C) (D)

2.
The picture shows how a shape can be 'turned' or 'flipped'.



Brian made copies of this shape.

To make this design he turned some shapes once, he flipped some shapes once and some shapes he left the same.



How many shapes were flipped?

(A) 3 (B) 5

(C) 6 (D) 7

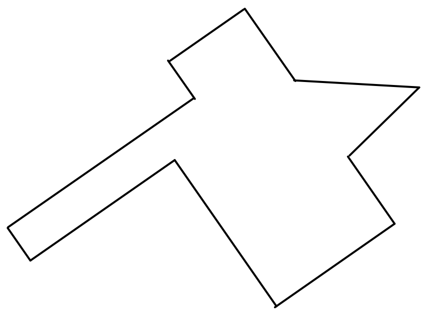
(2) Breaking down complex shapes into their basic components

The grade 5 teachers commented that in grade 5 they teach areas of squares and rectangles, and not a combination of shapes given in a drawing. They agreed that it would be a good idea to use simple combinations of shapes to teach area and perimeter, so that learners can be given opportunities to see shapes in different combinations and orientations. It is also important for learners to realise that we can find the area of any shape – not only squares and rectangles.

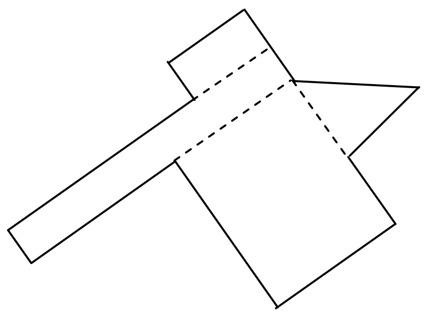
The two grade 5 items involve working with square shapes which are oriented and combined in various ways to produce a more complex shape. Learners have to keep in mind all the way through solving the problem that they need to work with both separate and combined aspects of the complex shapes. For example, in the second problem, it is easy to forget while calculating the area of the large square, that there are four other small squares that need to be included in the final answer.

Teachers of grade 4 and upwards can incorporate working with shape combinations as they work with other aspects of shape. This can be done in two ways. First, learners can make their own shape combinations and explain how they built up the complex shapes they made. Second, learners can be given pictures of complex shapes and asked to break them down into two or more basic shapes by drawing dotted lines across the given shape:

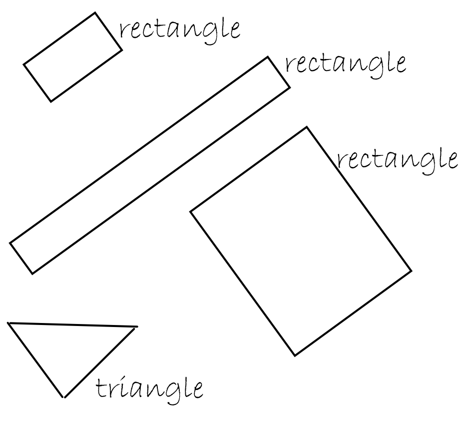
For example, learners can be asked to break down the complex shape drawn below into squares, rectangles and triangles. They can draw each shape separately into their books in the same orientation as it is found in the complex picture and label each shape:



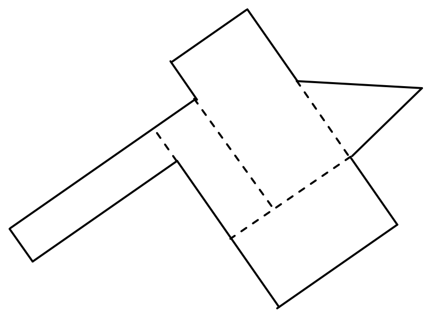
Can be broken down as follows:



And its separate shapes drawn and labelled as:



There are many possible ways to break down this shape. For example, it could also be "broken down" as:



Most classes will find it very interesting to see how many different ways are possible, first as a small group exercise, and then coming together as a class, to discuss different solutions. More examples of working with shape orientations and combinations are given in activity sheets 1 and 2 at the end of this newsletter, and we also discuss "playing" with shape in more detail in the next section.

(3) Working with grids that are not made up of unit squares

We are now entering into mixing LOs 3 and 4, where shape and measurement are combined in problems. The teachers who worked on the ICAS items said that even in Grade 6, where simple shape combinations are used in problems, teachers tend to only use $1cm^2$ grids. They suggested that in order to help learners to know how to deal with problems involving complex shapes, grade 6 learners can begin to discover that areas do not necessarily need to be measured using only $1cm^2$ grids. Firstly, make sure that learners know the terms being used, such as "1cm by 1cm", or "2cm by 2cm", "grids", "$1cm^2$", "$4cm^2$", and "area", so that they can work with these terms when making new discoveries.

Learners can start by finding out about how to use 2cm by 2cm grids. Provide learners with a square that is 2cm by 2cm, but do not tell them its dimensions. Then ask them to place a 1×1 cm grid onto the square and tell you what they notice about it. They need to be able to tell you that there are four 1cm^2 squares inside the big square. Let them talk to you about the big square being "2 by 2 cm" and its area being 4 cm^2. Also tell them that they do not always have to use a 1cm^2 grid to work out the areas of shapes. Now give them a ready-drawn 2cm by 2cm grid. Ask them about a single square in the grid. Encourage them to make the link between the dimensions of the single 2cm by 2cm square and the fact that its area is 4cm^2, and each square in the grid being exactly the same. Ask them what the areas of two, three and four squares on the 2×2 grid will be, to see if they understand how a 2×2 grid works. Once you are satisfied that they can work with the 2×2 grid on its own, you can give them easy shape combinations, and they can calculate the areas of these by first using their new 2×2 (or 4cm^2) grids and check their answers with the more familiar 1cm^2 grid.

You can do similar activities with 3cm×3cm and 5cm×5cm grids, and this will help your learners to consolidate their understanding of area by using different units to measure area. This kind of activity will help learners not to have the notion that all squares are 1cm by 1cm – especially when working with complex shapes.

The suggestions that were made about working with the above three areas of difficulty should prepare learners for working with a combination of spatial and measurement problems. If learners are comfortable working with different shape orientations and combinations of shapes, then calculating areas of these shapes using grids of 1cm×1cm, 2cm×2cm, etc will be challenging, but can be accomplished. Also, later, when they need to apply area and perimeter formulae to complex shapes, they will know how to visualise the shapes to decide which formulae they can use when.

MORE ABOUT WORKING WITH SPATIAL REASONING AND MEASUREMENT IN PRIMARY SCHOOL

Since so many learners find working with space and shape concepts difficult, they should work with physically manipulating objects as much as possible – especially in the lower grades. Cutting out paper shapes and fitting them together to produce complex shapes, drawing on squared paper, and experimenting with shapes will help them later to mentally picture these same shapes and how they may be manipulated.

Experience with tangram puzzles can also be helpful because learners can use the puzzles to build new shapes from old shapes. Shapes may also be used to make interesting pictures. All this sort of "play" helps learners to become more familiar with shapes. Three worksheets are provided to introduce working with shape in a fun way, at the same time as introducing important spatial concepts to the learners. Activity sheets 2, 3, and 4 are examples of how these activities may be done.

Levels of spatial reasoning

Learners' understanding of the increasingly complex relationships between components of geometric figures and between different geometric figures is crucially important. They should do activities in which they work with spatial reasoning in different ways.

Some interesting work was done by Pierre van Hiele and his wife, Dina van Hiele-Geldof in the 1950's, with respect to how spatial reasoning with geometric figures develops. They proposed that spatial reasoning can be classified into levels[2]. We discuss these levels briefly in this newsletter, because understanding how spatial reasoning develops will help teachers to address learners' weaknesses with respect to spatial visualisation, thinking and reasoning.

Pierre van Hiele argues, along with Jean Piaget, that one "must provide teaching that is appropriate to the level of children's thinking[3]". Learners cannot deal with solving problems using complex properties of shapes if they have not understood and worked competently with lower levels of spatial reasoning. We encourage you, therefore, to go back to basics, if you need to, using what we suggest below, to help your learners to overcome difficulties with spatial reasoning.

The van Hiele levels of geometric thinking can be described as follows: Level 0 is a *visual level*, where learners recognise shapes because of their appearance. They do not necessarily verbalise their visual recognition of the shape. For example, a five year old child might show you all the triangles in a set of shapes because s/he recognises that they are triangles because they "look the same"– not necessarily because they have three sides.

Level 1 is a *descriptive level*, where language is necessary to describe the properties of different shapes. For example, a triangle has three sides; a rectangle has four right angles; a circle always has the same distance from the centre to the edge; and so on. Learners do not relate different properties of the same figure to each other. For example, they will not realise that a rectangle with four right angles also has two pairs of opposite sides equal.

At level 2, the *informal deduction level*, logical relationships between properties of shapes are described. For example, a rectangle has two pairs of parallel sides, which means that the same two pairs of sides must have equal lengths, in order for them both to be parallel. This level is essential for higher levels of deductive reasoning to develop. Being able to relate all the properties of a particular shape to each other allows the learners to move on to relating properties of different shapes to each other.

For example, all squares are also rectangles because they have four right angles, and two pairs of opposite sides parallel and equal in length; but all rectangles are not squares because all rectangles do not have all four sides equal in length. All rectangles are also trapezia because all rectangles have at least one pair of opposite sides parallel; but all

---

[2] An excellent summary in mind map form of the van Hiele levels and what they entail cognitively can be found at: http://agutie.homestead.com/FiLes/mindmap/van_hiele_geometry_level.html

[3] You can read a full discussion of the levels, together with suggested activities to aid spatial understanding in the following reference:
van Hiele, P. M. (1999). Developing geometric thinking through activities that begin with play. *Teaching Children Mathematics. 5(6)*, 310–316.

trapezia are not rectangles because trapezia do not have all the other characteristics of rectangles mentioned above.

Van Hiele described two other levels, but these are more likely to be used in high school and will be discussed in Part 3b of this newsletter.

Teaching for spatial reasoning

In their earlier research the Van Hieles argued that if a learner is located at a particular level, s/he will not understand a teacher who teaches at a higher level. For example, consider the following common classroom situation: Some grade 7 learners are conceptually at level 1, knowing how to recognise different quadrilaterals and what some of their properties are, but have not realised that a rectangle can be described according to its two pairs of equal sides **and** parallel sides, **and** right angles, **and** equal diagonals. The learners probably also do not know how to relate and compare these properties to each other for different shapes.

The teacher wants them to compare the properties of sides and angles of parallelograms and rectangles to deduce why a rectangle is a special kind of parallelogram, which is included in LO3 for grade 7. She expects them to be able to do this, and becomes frustrated because they seem unable to understand her reasoning why all rectangles are parallelograms, but that all parallelograms are not rectangles.

But the learners are at level 1, and have not yet realised that they can describe rectangles by comparing their properties with those of parallelograms. The teacher needs to understand that they are conceptually at level 1 and work out how to bring them to level 2. Then they can make connections between the properties of the sides and angles of rectangles and parallelograms (separately), and then compare their properties with each other.

The van Hieles suggest strategies for taking learners through the levels of spatial thinking, to result in their being successful spatial reasoners. Advancement through the levels is not dependent on age or maturity, but rather on competency in each level before the next level is developed. Some of the van Hiele's suggestions are described below. Some practical examples you can use in the classroom are in the activity sheets at the end of this newsletter.

Developing deeper level 0 understanding

The van Hieles suggest that when approaching a new concept in any grade at any level of spatial reasoning, teachers and learners can observe and discuss the objects of interest. Learners should be encouraged to explain their arguments concisely using correct language. Level-specific vocabulary is introduced, such as "sides", "faces", '"round", "straight", and the names of basic shapes, such as square, rectangle, circle, oval, crescent, hexagon, and so on, if the learners are at level 0.

The levels of spatial reasoning are not age specific. Therefore, teachers need to be aware of what level the learners are at and teach them to move from this level to the next, independent of the age or grade of the learners. Teachers can deal with different spatial orientations whenever they teach new concepts, because learners need to recognise and discuss shapes by observing their characteristics in any orientation. For example, learners should understand early when working with shape, that a square rotated to sit on its vertex is still a square, and that its vertex is still 90°. A triangle that is not equilateral in shape is still a triangle.

The van Hieles suggested some activities through play that will greatly facilitate learner geometrical thinking at level 0, and help them to move to level 1. Note that "play" is not for its own sake, but is focused and directed towards extending understanding in increasing levels of complexity working with spatial problems. Look at Activity sheet 2, for an example of how this level of spatial reasoning may be approached in teaching.

Developing deeper level 1 understanding

If learners are at level 1, teaching new material involves learners working through materials carefully sequenced by the teacher, directed to gradually reveal the structures that are characteristic in preparation for level 2. The focus is on finding out the properties of different shapes. This includes:

- shifting from simple identification to properties, by using concrete or virtual models to define, measure, observe, and change properties
- using models and/or technology to focus on defining properties, making property lists, and discussing sufficient conditions to define a shape
- doing problem solving, including tasks in which properties of shapes are important components
- classifying using properties of shapes[4].

For example, learners may be discovering all the properties of quadrilaterals for the first time. Then they might work with concepts and terms such as "perpendicular", "properties", "intersection", "vertex/ vertices", "diagonal", and "classify", also through observation and discussion of the objects of interest.

Learners may be asked to compare quadrilaterals through guided activity and manipulation of shapes. Using cardboard cut-outs of all shapes and sizes of quadrilaterals, learners can sort shapes; explaining in as much detail as possible why they have chosen to sort them in that particular way. Directed questioning can encourage learners to change their minds about why shapes should be grouped in certain ways. Look at Activity sheets 3 and 4 and note how these activities are practical examples of the above principles.

Developing deeper level 2 understanding

As learners begin to be more familiar with different properties of shapes in level 2, the teacher can encourage them to express and exchange their emerging views about how properties relate to each other. She encourages conversation, using accurate and appropriate vocabulary. She may raise questions such as: Is a rectangle a square? Why?; Is a square a rectangle? Why?; or Why are those triangles congruent?". Part of instruction at level 2 includes:

[4] This information was obtained from http://images.rbs.org/cognitive/van_hiele.shtml

- doing problem solving, including tasks in which properties of shapes are important components
- using models and property lists, and discussing which group of properties constitute a necessary and sufficient condition for a specific shape
- using informal, deductive language ("all," "some," "none," "if-then," "what if," etc.)
- investigating certain relationships among polygons to establish if the converse is also valid (e.g., "If a quadrilateral is a rectangle, it must have four right angles; if a quadrilateral has four right angles, must it also be a rectangle?")
- using models and drawings (including dynamic geometry software) as tools to look for generalizations and counter-examples
- using properties to define a shape or determine if a particular shape is included in a given set[5].

Summary

What does this discussion mean for teaching spatial relationships in primary and high school? A large amount of time can be spent by learners on manipulating shapes and making findings explicit – learners can explaining their discoveries to each other or formalise their findings in written summaries. Teachers need to allow time to be spent on these activities, in alignment with the NCS, where learners work together on well-planned and scaffolded activities. Teachers need to focus on planning lessons that allow learners to work a lot with shapes and their properties in different orientations and combinations. For example, a square is placed next to a triangle, and questions asked about the properties of both figures when compared to each other (see activity sheets 3 and 4).

The concepts learned in LO3 are used in many other areas in mathematics. For example, it is likely that learners will have to incorporate knowledge from LO3 to solve problems in LO4, so it is a good idea to teach LO3 ideas so that it is a scaffold for LO4 work. In addition, working with shapes and combinations of shapes in different orientations is important for the following reasons: it helps learners to later visualise the shapes making up 3–D objects, makes visualising 3–D nets easier, and later helps them when solving problems in co-ordinate geometry and applied mathematics, such as maximising or minimising dimensions of objects in differential calculus.

In summary, plan learner activities carefully, so that they are scaffolded to help the learner to build competence in each level, and move to the next level. Allow learners to work through finding relationships and geometrical structures, rather than informing them of those relationships through extensive lecturing and notes. Help learners to move into new levels starting at the level they are currently at. Focus on mental orientation problems while learners are in levels 0 – 2 of geometric reasoning.

Lynne (project manager), Mavis, Berdinah and Manono (grade 3) discussing the grade 3 test

[5] Teaching learners at higher levels of spatial reasoning is more appropriate in the high school, and will be discussed further in issue 3b of this newsletter.

SOME LEARNER ACTIVITIES

The activities described in the activity sheets below come from Pierre van Hiele's suggestions in the journal *Teaching Children Mathematics*. He works with a "mosaic pattern", drawn for you in Activity sheet 2 below. He has designed activities with the pattern to teach children how to reason spatially. The activity sheets describe some ways in which the mosaic pattern can be used to teach spatial reasoning.

Mmatladi (grade 5 team leader) doing some paper folding while planning a spatial reasoning lesson with her group

Using the Activity Sheets

Except for Activity sheet 1, these activities are not for teachers to copy and hand out to learners. They exist as a set of instructions and suggestions to teachers about how questions and tasks can be used with learners. The tasks are in between the instructions, so you will have to cut them out and make them into worksheets for learners. Please read through these and plan how you will use the activities with respect to the amount of class time you have and according to the levels of your own learners.

Peter, Steven and Siphiwe (grade 5) planning lessons

The mosaic provided in Activity Sheet 2 is sized so that you can copy it directly onto cardboard multiple times and let your learners have their own mosaics to experiment with. If they keep their mosaics safely in a folder they can use the mosaic in many different ways to develop spatial reasoning, in the ways suggested in the activity sheets.

Nosisa and Nicol (grade 5) planning a grade 5 problem solving lesson

SHAPES ACTIVITY SHEET 1

Finding the shapes (grade 4 upwards)

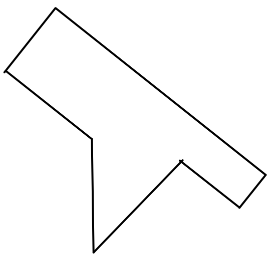
Use a ruler to draw lines inside each shape given below to break it down into separate shapes.

(1) Break down each complex shape into individual shapes in two different ways.

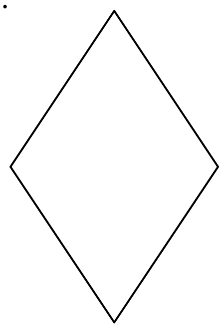
(2) Draw each separate shape in the same orientation as you see it in the original shape and name each separate shape (If you cannot name it then you have to choose another way to break down the shape).

(The first one has been done for you on the next page).

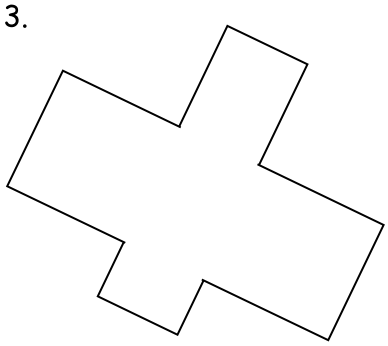
1.



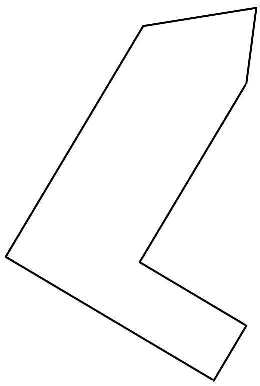
2.



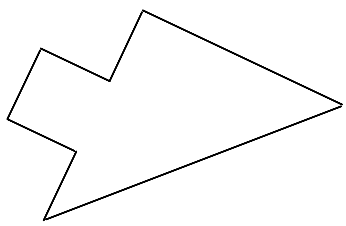
3.



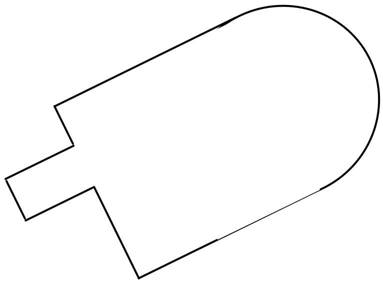
4.



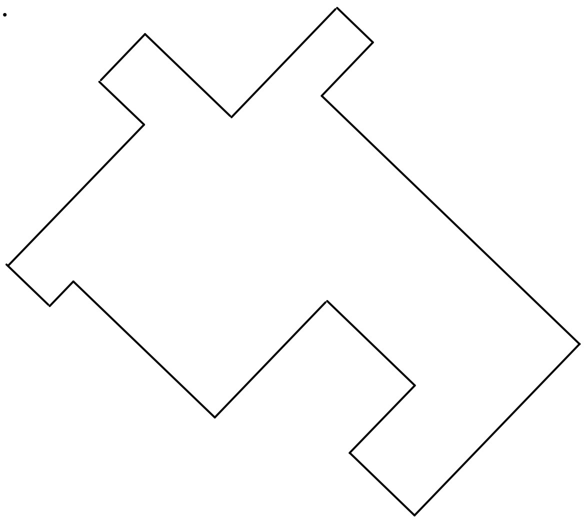
5.

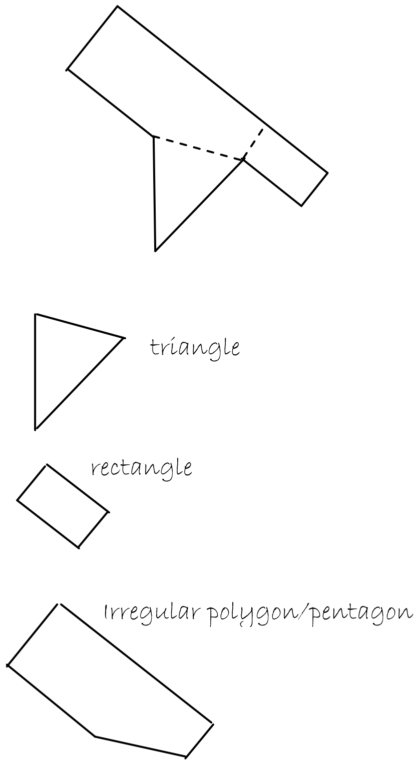
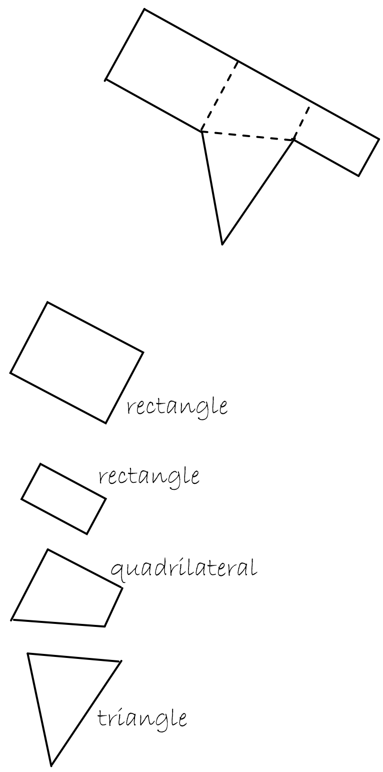


6.



7.

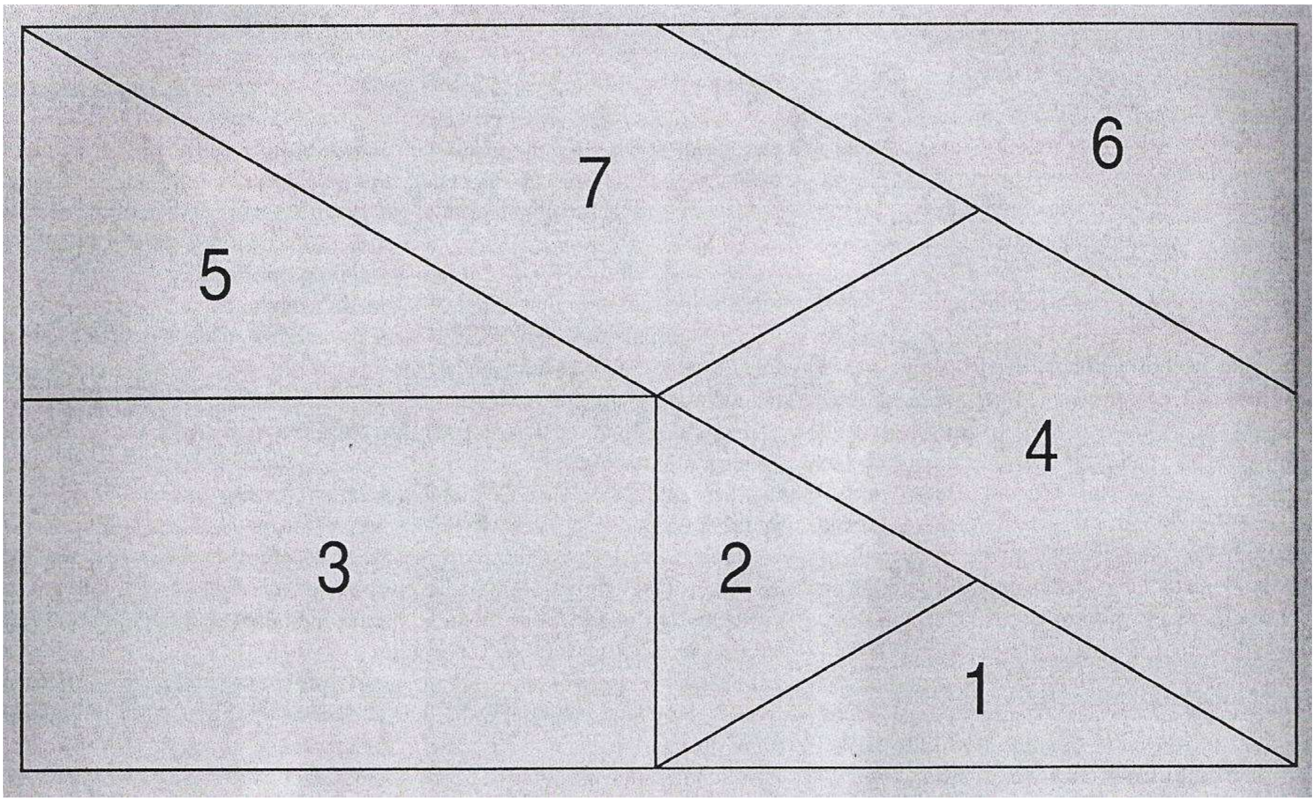


Method 1:	Method 2:
	

SHAPES ACTIVITY SHEET 2

Level 0 (grades 1 to 3)

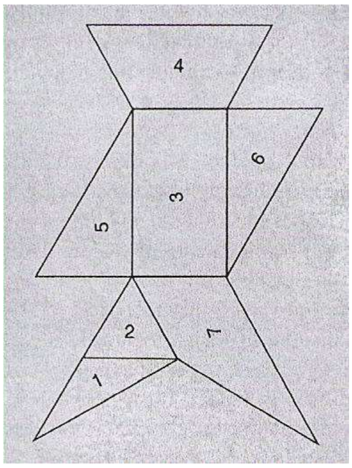
1. The shape below is the "mosaic" pattern used by van Hiele in his article in *Teaching Children Mathematics*. It is similar to the tangram you use at school. Below it are some ideas for teaching spatial reasoning to primary school learners using the mosaic pattern. The suggested activities are essential as beginner activities to help learners to discover for themselves and discuss aspects of geometric shapes at a visual level (level 0 – see page 9). These activities help learners to discover how the mosaic shapes may fit together or not, and how the appearances of shapes differ. The activities also help learners to see the shapes in different orientations. Orienting the shapes differently results in different pictures being designed, and also helps learners to realize that shapes do not have to be drawn in one way only. Learners develop two-dimensional spatial awareness by doing these activities, so do not cut short the essential time needed for them to work with the activities.

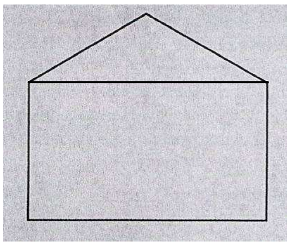


This shape should be used exactly as it is given here. Make copies of it, including the numbers, onto cardboard and let learners cut out the pieces. The geometric shapes on the large rectangle are as follows: isosceles triangle (piece 1), equilateral triangle (piece 2), rectangle (piece 3), isosceles trapezoid (piece 4), right-angles triangles (pieces 5 and 6), and irregular trapezoid (piece 7). The learners do not need to know the names of the individual pieces to do this activity, but if they are in grade 5 or upwards they need to know the shape names.

Learners should work individually, but allow them to share ideas and talk while they are working with their own shapes. They may talk about each others' shapes. This is acceptable, as they are still learning how the shapes relate to each other. Give them lots of time to discover the properties of the different shapes in the mosaic.

a) What pictures can they make from putting the pieces together? (Some learners will make complex pictures and others will make simpler ones. Also they could make abstract pictures). For example, says van Hiele, one could make a person (more complex) or a simple house, as shown below:





Let learners trace around their pictures into their books or onto paper, and colour them. It is a fun activity and can be pasted onto the wall so that they can see what others have drawn.

Other activities learners can do with the mosaic after the pieces have been cut out and separated:

b) (i) If you put two mosaic pieces together, which pieces will have their edges matching exactly (no parts of the edge sticking out)?

(ii) Do all of the edges of the pieces you are looking at match exactly? Why/why not?

c) (i) Every piece, except for 1 and 2, can have two of the other pieces put exactly on top of it. Find all the possible combinations where two pieces can be put on top of a single piece. To start you off, pieces 5 and 6 can be put exactly on top of piece 3.

(ii) Can we have other combinations if some pieces are turned upside down, so that their numbers are not showing?

d) (i) Can every piece be turned upside down where the number is not showing, and be <u>exactly</u> the same whichever side is showing?

(ii) Which pieces are exactly the same if they are turned upside down?

(iii) Which pieces are not exactly the same if they are turned upside down? Why/why not?

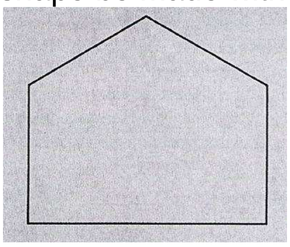
e) Are there any pieces which can have three other pieces put exactly on top of it. (There is one piece that can).

f) Now that you have cut out the individual pieces, can you put the pieces back together to make the original rectangle? (This is for grade 5 and above).

2. The mosaic can also be used to follow specific task instructions. Learners can be put into groups of 3 or 4 and given a task card. They can swop task cards once they have completed the task successfully. (This activity is also taken from van Hiele's article in *Teaching Children Mathematics*.) You can photocopy these and put them onto card, and laminate the cards, so that they are more durable.

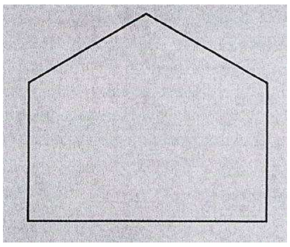
House Puzzle

1. On a piece of paper make a house like this one with two pieces.
2. Trace around the whole house to make a shape.
3. Make this shape with two other pieces.
4. Make this shape with three pieces. Can you find two ways to do it?
5. Can this shape be made with four pieces?



Tall-House Puzzle

1. The picture below shows a short fat house. On a piece of paper, make a tall thin house with piece 2 as the roof and one other piece.
2. Trace around the house you made.
3. Make the same tall house shape with pieces 5 and 7.
4. Can the tall house be made with three pieces?

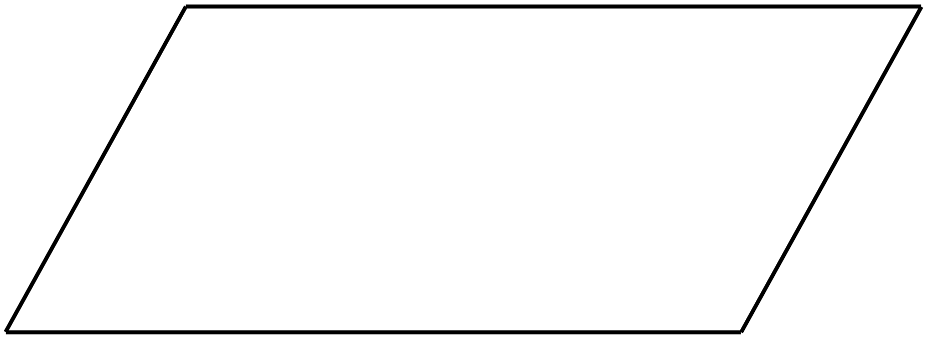


Make a Puzzle

1. Use any two, three or four pieces. Make a shape. Trace around it on a piece of paper. Colour it.
2. Can you make this shape with other pieces?
3. Write a name and title for your puzzle

Parallelogram Puzzle

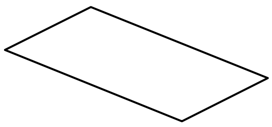
Make this shape using two or more pieces from the mosaic. There are 4 different ways to make the shape. Find them all. (Use this exact shape. It will fit the cut-outs from the mosaic)



Rectangle Puzzle

1. A rectangle exactly the same as piece 3 of the mosaic can be made with two other mosaic pieces.
2. A rectangle exactly the same as piece 3 can also be made using 3 other pieces of the mosaic
3. Make both of the rectangles identical to piece 3.

Two different parallelograms

A parallelogram has a shape like this: 

1. Make a parallelogram with pieces 2 and 4.
2. Make another different parallelogram using pieces 7 and 1.

Describing triangles (1)

Pieces 5 and 6 are the same.

1. Hold up piece 5 or 6, close your eyes, and run your fingertips over the corners of the shape.
2. With your eyes closed describe to your group each corner of the triangle. Is it the same as or different from the other corners? How is it the same or different?
3. Make a triangle that looks the same as piece 5, but is bigger using **two** pieces
4. Make a triangle that looks the same as piece 5, but is bigger using **three** pieces
5. Make a triangle that looks the same as piece 5, but is bigger using **four** pieces. You can do this in two different ways. Find both ways.

Describing triangles (2)

1. Put piece 2 onto your page and draw around it. Onto this shape:
2. Make a triangle that is **the same shape** as piece 2, but a different size, using 2 pieces
3. Make a triangle that is **the same shape** as piece 2, but a different size, using 3 pieces

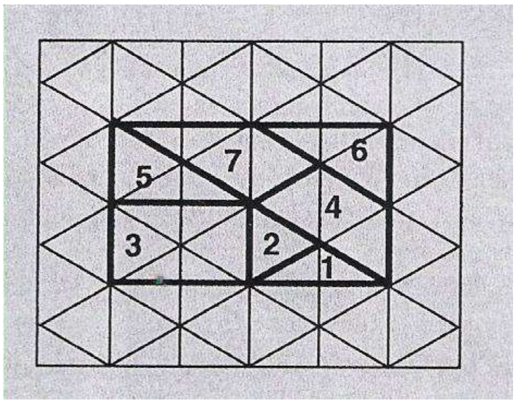
SHAPES ACTIVITY SHEET 3:

Equilateral Triangles

Level 1 (approximately grades 4 and 5)

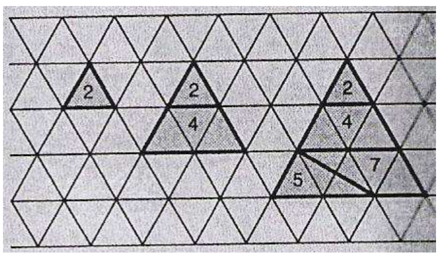
The following activity deals with further discovering properties of geometric shapes and naming the shapes. Please note that this is only an example, and is not complete enough to make sure that all geometric reasoning at level 1 is addressed by following this activity.

The whole large rectangle mosaic, broken up into shapes 1 to 7, can be placed onto a grid of equilateral triangles, as shown below. Make a similar grid for yourself and one per pair of learners. Make the grid using triangles the same size as piece 2. Give each learner an extra piece 2 of the mosaic.



Work first with piece 2. Learners already know that it is a triangle.

1. On the 12 by 12 grid of equilateral triangles, learners should make another triangle that is exactly the same shape, but is a different size, by joining pieces 2 and 4.
2. Then they should make another triangle that looks the same as the first two triangles, but is even bigger than the one made in question 1, using pieces 2, 4, 5 and 7. (For the teacher's benefit: Look at the grid below to see how each triangle is made from pieces of the mosaic)



3. Learners should put all three differently-sized triangles next to each other onto the triangle grid (they can do this because they have enough mosaic pieces when working in pairs) and answer the following questions:

   (i) Compare the sides of the smallest triangle with each other. How do they compare with each other?

   (ii) Compare the sides of the second biggest triangle with each other. How do they compare with each other?

   (iii) Compare the sides of the biggest triangle with each other. How do they compare with each other?

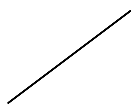
At this point learners can be introduced to the concept of "equilateral", meaning equal sided – because they will see that for all triangles, no matter their sizes, all three sides have equal length.

4. (i) Compare the sides of all the triangles. How are the sides different for the three triangles?

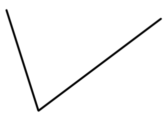
   (ii) Look at the corners of all three triangles. Describe the corners of each triangle. How do the corners of the triangles compare with each other?

You need to help learners to realize that the "corners" of the triangles are all the same – regardless of the size of the triangle. Ask them why they can make the conclusion. Tell them to place a "corner" of each of the three triangles on top of each other and see for themselves that they fit on top of each other exactly. It should be apparent to them that the size of the angle is not dependent on the size of the equilateral triangle.

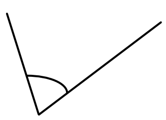
They do not necessarily know what vertices and angles are. If they are in grade 6 upwards, at this point you can tell them that the corners are called vertices and the amount of "space" between one "arm" and the other (you can draw this for them as you explain) is called an angle. **Note:** do not draw any of the arms of the angle horizontally as you explain what an angle is. Draw a ray:



Draw another ray:



And then show where the angle may be found:



Note: A new term to teach learners is "angle"

5. Let learners work with piece 2 of the mosaic. They are to work with some more properties of equilateral triangles, in a transition from level 0 to level 1 of spatial reasoning. Instruct them to do the following two activities:

   (i) Flip piece 2 upside-down, so that the number is facing down. Put the upside-down triangle onto the equilateral triangle grid. Ask learners if the triangle looks different in any way now that it has been flipped, on the grid. (They should see that there is no difference)

   (ii) Colour in one of the vertices of piece 2. Place it onto the triangle grid and note where the coloured vertex is. Then rotate the triangle on the grid without moving it sideways or up and down, so that the coloured vertex is in a new position. Apart from the coloured vertex being in a new position, is there anything different about the appearance of the triangle? (Learners should note that it appears exactly the same; even though it has been rotated.

   Note: new terms introduced in this activity to grade 5 and upwards include "rotate", "vertex"

Help your learners to observe two important properties of equilateral triangles:

1) rotational symmetry, and
2) flip symmetry (the same shape if it is number-up or number-down)

Comments on the activity:

Why does this activity lend itself to development of level 1 spatial reasoning?

Learners need to relate their general experiences of the shape of equilateral triangles (level 0) to their properties (level 1). By the end of these activities they should understand that equilateral triangles are triangles that specifically have

- all three sides equal in length
- all three angles the same, regardless of the size of the triangle
- identical appearance when flipped number-downwards.
- rotational symmetry – related to the fact that all the angles are the same

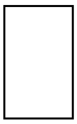
These properties will be discussed again in activity sheet 5

SHAPES ACTIVITY SHEET 4:

Rectangles

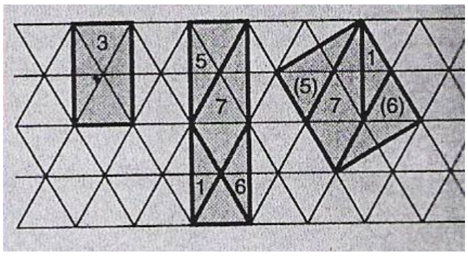
Level 1 (Approximately grades 4 and 5)

This activity can also be done in pairs, as duplicates of mosaic pieces are needed. Put piece 3 of the mosaic on the triangle grid in the following orientation:



Learners can be told that piece 3 is a rectangle. If they already know the term, ask them why it is a rectangle. They may know that it is but struggle to explain why.

1. Onto the same triangle grid, help learners to build another two rectangles with pieces 1, 5, 6, and 7 in two different arrangements, as shown in the picture below:



   The first arrangement makes a tall rectangle, and the second makes a skew rectangle – both of which are important for showing learners that rectangles may be different sizes and oriented in different ways.

2. Ask learners to see if they can build other rectangles with the mosaic pieces. Give them time to experiment and to give them enjoyment discovering new combinations. (The most difficult rectangle to make would be to reproduce the original large mosaic rectangle. Give them a chance to see if they can place the pieces to make the original large rectangle).

Note: a square can also count as a rectangle and you should help your learners to see this

- Giving learners time to play with rectangles is important because with time they will notice that rectangles do not always look the same. Sometimes they can be long and thin, sometimes short and fat, and sometimes square.
- A large rectangle is not necessarily an enlargement of a smaller rectangle, but for equilateral triangles, large equilateral triangles are exact enlargements of smaller ones. Ask learners what makes rectangles have these different appearances, whereas equilateral triangles always look the same even though some are big and others small.
- Ask learners to find examples of where they see rectangles in their everyday lives.
- Learners must note that for rectangles
    - All vertices are “straight” or “square”. (**Note:** the terms “right angle” and “perpendicular” should be introduced now, for grade 5 and upwards)
    - Rectangles do not have rotational symmetry

3. Ask learners to make up shape 3 using pieces 5 and 6. Introduce the term “right-angled triangle” and ask them why the right angled triangles match the rectangle so well. Learners need to recognize that right angles are also found in triangles. Ask learners to see if they can make shape 5 or 6 using combinations of other shapes. Can they make larger right-angled triangles (try 1, 2, and 5, or 3, 5 and 6 in different combinations). Are the large triangles exact enlargements of piece 5 (6)? Why / why not?

Two more activity sheets using van Hiele’s mosaic have been provided in issue 3 (b). These activity sheets teach learners to understand the concept of angles, which many find very difficult to understand; and use the mosaic to introduce the concepts of area and perimeter.