Mathematics for Teaching Matters[1]

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In this paper, I illuminate the notion of mathematics for teaching (its matter) and argue that it matters (it is important), particularly for mathematics teacher education. Two examples from studies of mathematics classrooms in South Africa are described, and used to illustrate what mathematics teachers use, or need to use, and how they use it in their practice: in other words, the substance of their mathematical work. Similarities and differences across these examples, in turn, illuminate the notion of mathematics for teaching, enabling a return to, and critical reflection on, mathematics teacher education.

## Introduction

This paper explores the notion of *mathematics for teaching,* and why it matters for the teaching and learning of mathematics in general, and mathematics teacher education in particular. This exploration builds on the seminal work of Lee Shulman. In the mid-1980s Shulman argued cogently for a shift in understanding, in research in particular, of the professional knowledge base of teaching. He highlighted the importance of content knowledge in and for teaching, criticising research that examined teaching activity without any concern for the content of that teaching. He described the various components of the knowledge base for teaching, arguing that content knowledge for teaching included subject matter knowledge (SMK), pedagogical content knowledge (PCK) and curriculum knowledge (Shulman, 1986; 1987). Shulman's work set off a research agenda, with a great deal focused on mathematics. This paper draws from and builds on the mathematical elaboration of Shulman's work.

The profound insight of Shulman's work was that being able to reason mathematically, for example, was necessary but not sufficient for being able to teach others to reason mathematically. Being able to teach mathematical reasoning involves recognising mathematical reasoning in others' discourse, and at various curriculum levels, being able to design and adapt tasks towards purposes that support mathematical reasoning, and critically working with or mediating the development of such in others. We could say the same for being able to solve algebraic or numeric problems. Most mathematics teachers and mathematics teacher educators would agree with this assertion. Yet, in the particular case of mathematical reasoning, its actuality in curricular texts, classroom practices and learner performances remains a challenge in many, if not most, classrooms (Stacey & Vincent, 2009). We could say the same for learner performance in many areas of mathematics, as well as algebra. Despite the longevity and consistency of elementary algebra in school mathematics curricula worldwide, large numbers of learners experience difficulty with this powerful symbolic system (Hodgen, Kuchemar, Brown & Coe, 2008).

In this paper I argue that strengthening our understanding of the mathematical work of teaching, what some refer to as *mathematics for teaching,* is a critical dimension of enhancing its teaching and learning. Mathematics for teaching matters, for all our

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learners, as do its implications for mathematics teacher education. I will develop this argument through examples from school mathematics classrooms that invoke mathematical reasoning, together with comment on developments in mathematics teacher education in South Africa. Ultimately, the argument in this paper poses considerable challenges for mathematics teacher education.

## Teaching and learning mathematics in South Africa

Post-apartheid South Africa has witnessed rapid and intense policy and curriculum change. New mathematics curricula are being implemented in schools across Grades 1–12, where there is greater emphasis than before on sense-making, problem-solving and mathematical processes, including mathematical reasoning, as well as on new topics related to data handling and financial mathematics. New education policy and curricula have strong equity goals, a function of the deep and racialised inequality produced under apartheid that affected teachers and learners alike. New policies and qualifications have been introduced into teacher education, with goals for improving the quality of teachers and teaching. In the case of mathematics, there is also a quantitative goal – of need to address enduring critical shortages of qualified secondary mathematics teachers. Tertiary institutions have responded, offering new degree and diploma programs for upgrading teachers in service, retraining into teaching, and preparing new teachers.

It is in moments of change that taken-for-granted practices are unsettled, in both inspiring and disconcerting ways. Moments of change thus provide education researchers and practitioners with challenging opportunities for learning and reflection. Of pertinence to this paper is that the challenge of new curricula in schools and thus new demands for learning and teaching, on top of redress, bring issues like the selection of knowledges for teacher education development and support to the fore. Mathematics teacher educators in all tertiary institutions have had the opportunity and challenge to make decisions on what knowledge(s) to include and exclude in their programs, and how these are to be taught/learnt. This has meant deliberate attention to what mathematics, mathematics education and teaching knowledge teachers need to know and be able to use to teach well. This is no simple task: in South Africa, teaching well encompasses the dual goals of equity and excellence. At the same time as strengthening the pool of mathematics school leavers entering the mathematical sciences and related professions, high quality teaching also entails catering for diverse learner populations, and inspiring school learners in a subject that all too often has been alienating.

*Hence the question: what selections from mathematics, mathematics education and teaching*[2] *are needed to provide the greatest benefit to prospective and in-service teachers?*

Shulman's categories provide a starting point to answering this question. Others, particularly Ball and her colleagues working on mathematical knowledge for teaching in Michigan USA, have argued that these categories need elaboration; and that elaboration requires a deeper understanding of mathematics *teaching*, and hence, of teachers' mathematical work. Ball, Thames and Phelps (2008) have elaborated Shulman's

[2] Mathematics education here refers to the field of research and other texts related to mathematics curricula; teaching refers to the professional practice.

categories, distinguishing within subject matter knowledge, between *Common* and *Specialised Content Knowledge* where the latter is what teachers in particular need to know and be able to use. Within Pedagogical Content Knowledge, they distinguish *knowledge of mathematics and students*, and *knowledge of mathematics and teaching*. These latter are knowledge of mathematics embedded in (and so integrated with) tasks of teaching, that is, a set of practices teachers routinely engage in or need to engage in. In their more recent work where they examine case studies of teaching, Hill, Blunk, Charalambos, Lewis, Phelps, Sleep and Ball (2008) note that while their elaboration is robust, compelling and helpful, they underestimated the significance of what Shulman identified as Curriculum Knowledge. What this reflects is that all teaching always occurs in a context and set of practices, of which curricular discourses are critical elements. Ball et al.'s elaboration of Shulman's categories is useful, particularly as it has been derived from studies of mathematics classroom practice. They provide a framework with which to think about and make selections for teacher education. At immediate face value, they suggest that mathematical content in teacher education and for teaching requires considerable extension beyond knowing mathematics for oneself.

I go further to say we need to understand what and how such selections take shape in mathematics teacher education practice. As in school, teacher education occurs in a context and set of practices, and is shaped by these. In addition, as intimated above, in mathematics teacher education, mathematics as an "object" or "focus" of learning and becoming, is integrated with learning to teach. The research we have been doing in the QUANTUM[3] project in South Africa (that now has a small arm in the UK) has done most of its work in teacher education as an empirical site, complemented by studies of school mathematics classroom practice. The goal is to understand the substance of opportunities to learn mathematics for teaching in teacher education, and how this relates to the mathematical work teachers do in their school classrooms.

In this paper, I select two examples from studies of mathematics classrooms in South Africa. I use these to illustrate what mathematics teachers use, or need to use, and how they use it in their practice: in other words, the substance of their mathematical work. Similarities and differences across these examples, in turn, illuminate the notion of mathematics for teaching, enabling a return to, and critical reflection on, mathematics teacher education.

## Designing and mediating productive mathematics tasks

### Example 1: Angle properties of a triangle

The episode discussed below is described in detail in Adler (2001)[4], and takes place in a Grade 8 classroom. This teacher was particularly motivated by a participatory pedagogy, and developing her learners' broad mathematical proficiency (Kilpatrick, Swaffold & Flindell, 2001). She paid deliberate attention to supporting her learners' participation in mathematical discourse (Sfard, 2008), which in practice involved

[3] For details on QUANTUM, a project focused on Qualifications for Teachers Underqualified in Mathematics, see Adler & Davis (2006), Davis, Parker & Adler (2007); Adler & Huillet (2008), Adler (2009)

[4] The focus of the study reported in Adler (2001) was on teaching and learning mathematics in multilingual classrooms. There I discuss in detail the learners' languages, and how and why talking to learn worked in this class. I have since revisited this data, reflecting on the teachers' mathematical work (see Adler, 2006).

having them learn to reason mathematically, and verbalise this. It is interesting to note that the empirical data here date back to the early 1990s and long before curriculum reform as it appears today in South Africa was underway.

As part of a sequence of tasks related to properties of triangles, the teacher gave the activity in Figure 1 to her Grade 8 class. The questions I will address in relation to this task are: What mathematical work is entailed in designing this kind of task, and then mediating it in a class of diverse learners?

**If any of these is impossible, explain why; otherwise, draw it.**

- ❖ Draw a triangle with 3 acute angles.
- ❖ Draw a triangle with 1 obtuse angle.
- ❖ Draw a triangle with 2 obtuse angles.
- ❖ Draw a triangle with 1 reflex angle.
- ❖ Draw a triangle with 1 right angle.

*Figure 1. A triangle task.*

The task itself evidences different elements of important mathematical work entailed in teaching learners to reason mathematically**.** Firstly, this is not a "typical" task on the properties of triangles. A more usual task to be found in text books, particularly at the time of the research, would be to have learners recognise (identify, categorise, name) different types of triangles, defined by various sized angles in the triangle. What the teacher has done here is recast a "recognition" task based on angle properties of triangles into a "reasoning" task (reasoning about properties and so relationships). She has constructed the task so that learners are required to reason in order to proceed. In so doing, she sets up conditions for producing and supporting mathematical reasoning in the lesson and related proficiencies in her learners. Secondly, in constructing the task so that learners need to respond whether or not particular angle combinations are "impossible" in forming a triangle, the task demands proof-like justification—an argument or explanation that, for impossibility, will hold in all cases. In this task, content (properties of triangles) and processes (reasoning, justification, proof) are integrated. The question, of course, is what and how learners attend to these components of the task, and how the teacher then mediates their thinking.

In preparation for this lesson and task, the teacher would have had to think about the mathematical resources available to this classroom community with which they could construct a general answer (one that holds in all cases). For example, if as was the case, learners had worked with angle sum in a triangle, what else might come into play as learners go about this task? What is it about the triangle as a mathematical object that the teacher needs to have considered and that she needs to be alert to as her learners engage in reasoning about its properties?

Before engaging further with the details of the teachers' mathematical work, let us move to the actual classroom, where students worked on their responses in pairs. The teacher moved between groups, probing with questions like: "Explain to me what you have drawn/written here?", "Are you sure?", "Will this always be the case?" She thus pushed learners to verbalise their thinking, as well as justify their solutions or proofs. I foreground here learners' responses to the second item: Draw a triangle with two obtuse angles. Interestingly, three different responses were evident.

- Some said, "It is impossible to draw a triangle with two obtuse angles, because you will get a quadrilateral." They drew the shape shown in Figure 2.

*Figure 2. Student drawing of a triangle with two obtuse angles*

- Others reasoned as follows: "An obtuse angle is more than 90 degrees and so two obtuse angles give you more than 180 degrees, and so you won't have a triangle because the angles must add up to 180 degrees."
- One learner (Joe) and his partner reasoned in this way: "If you start with an angle say of 89 degrees, and you stretch it [to make it larger than 90 degrees], the other angles will shrink and so you won't be able to get another obtuse angle." Their drawing is shown in Figure 3.

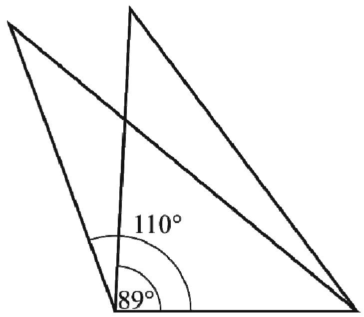



*Figure 3. Joe and his partner's response.*

The range of learner responses to this task is indicative of a further task-based teaching skill. The task is designed with sufficient openness, and so diverse learner responses are possible and indeed elicited. In addition, the third, unexpected, response produced much interest in the class, for the teacher, and for myself as researcher. The first two responses were common across learners and more easily predicted by the teacher.

Having elicited these responses, it is the teacher's task to mediate within and across these responses, and enable her learners to reason whether each of these responses is a general one, one that holds in all cases[5]. In the many contexts where I have presented the study and this particular episode, much discussion is generated both in relation to the mathematical status of the responses, and their levels of generality, as well as simultaneous arguments as to what can be expected of learners at a grade 8 level. What constitutes a generalised answer at this level? Are all three responses equally general? Is

[5] The interesting interactions that followed in the class are described and problematised in Adler (2001) and will not be focused on here.

Joe's response a generalised one? How does the teacher value these three different responses, supporting and encouraging learners in their thinking, and at the same time judging/evaluating their mathematical worth?

These are mathematical questions, and the kind of work this teacher did on the spot as she worked to *evaluate* and *value* what the learners produced was also mathematical work. The point here is that this kind of mathematical work i.e. working to provoke, recognise and then mediate notions of proof and different kinds of justification, is critical to effective teaching of "big ideas" (like proof) in mathematics. In Ball et al.'s terms, this work entails *specialised content knowledge* (judging the mathematical generality of the responses), knowledge of *mathematics and teaching* (designing productive tasks) and *mathematics and students* (and mediating between these and learners' mathematics).

We need to ask further questions about subject matter knowledge, or content in this example, and specifically questions about the angle properties of triangles. The insertion of a triangle with a reflex angle brought this to the fore in very interesting ways. Some learners drew the following, as justification for why a triangle with a reflex angle was possible; and so provoked a discussion of concavity, and interior and exterior angles.

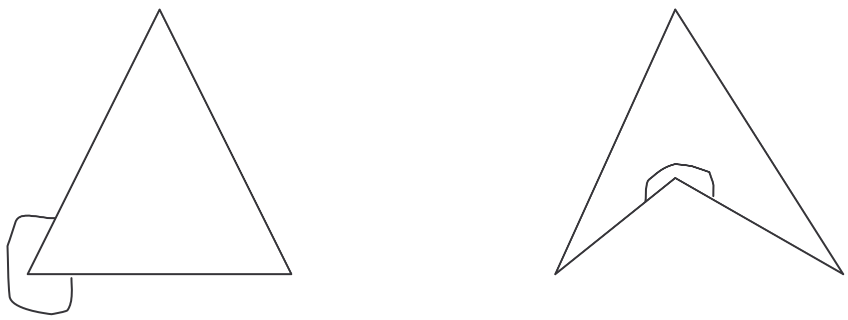


*Figure 4. Learner drawings to justify triangles with reflex angles.*

The tasks of teaching illuminated in this example are: task design where content (angle properties of triangles) and process (reasoning, justifying) are integrated; mediation of both mathematical content and processes; and valuing and evaluating diverse learner productions. The mathematical entailments of this work are extensive, and are illustrative of both subject matter knowledge and pedagogical content knowledge. The teacher here reflects a deep understanding of mathematical proof, and in relation to a specific mathematical object and its properties. To effectively mediate Joe's response and the two above, she would also need to ask suitable questions or suggest productive ways forward for these learners, so that their notions of proof and of the mathematical triangle are strengthened. Indeed, as learners in the class engaged with the second triangle drawn above, their focus was that the answer was incorrect because there were three reflex angles not one, and the teacher had a difficult time shifting them from this focus and onto the interior angles.

In Adler (2001), I show that as the teacher mediated the three different responses to the triangle with two obtuse angles, she worked explicitly to value each contribution and probe learner thinking. However, her judgment of their relative mathematical worth was implicit. She accepted the first two responses above, but probed Joe's, with questions to Joe that implied she was not convinced of the generality of his argument. I argued there that if teacher judgment of the varying mathematical worth of learner responses offered is implicit, it is possible that only those learners who can themselves make such judgements, or who are able to read the implicit messages in the teacher's actions, will appreciate and so have access to what counts mathematically. Sociological theory and empirical research inform us that these kinds of practices favour students with school cultural capital, and so can reproduce inequality. In Bernstein's (1996) terms, implicit practices will connect with learners who already understand the criteria for what are most legitimate responses; and alienate or pass by those who are not "in" the criteria. Typically these will be already disadvantaged learners (Parker, 2009).

The example here is compelling in a number of ways, and provokes the question: Where, when and how does a mathematics teacher learn to do this kind of work, and in ways that are of benefit to all learners? Before attempting to answer this and so shift back into teacher education, we need to look at additional and different examples of the mathematical work of teaching.

## Example 2: Polygons and diagonals — or a version of the "mystic rose"

The second example is taken from a Grade 10 class (see Naidoo, 2008), where the teacher posed the following task for learners to work on in groups: *How many diagonals are there in a 700-sided polygon?*

Here too, the teacher has designed or adapted a task and presented learners with an extended problem. They have to find the number of diagonals in a 700-sided polygon, a sufficiently large number to require generalising activity, and so mathematical reasoning. I pose the same questions here as for Example 1: What mathematical work is entailed in designing this kind of task, and mediating it in a class of diverse learners?
Many teachers will recognise the "mystic rose" investigation in this problem. The mathematical object here is a polygon and its properties related to diagonals. Yet the problem has been adapted from a well known (perhaps not to the teacher) mathematical investigation of points on a circle and connecting lines — a different, though related object. Here learners are not asked to investigate the relationship between the number of points on a circle and connecting lines, but instead to find an actual numerical solution to a particular polygon, albeit with a large number of sides and so approaching a circle. I have discussed this case in detail in Adler (2009), where I point out that unlike triangles and their properties, the general polygon and its properties is not an explicit element of the secondary school curriculum. However, the processes and mathematical reasoning required for learners to solve the problem are desired mathematical processes in the new curriculum.

My concern in this paper is not with the merits of the problem and its adaptation in an isolated way. Rather, I wish to reflect on the mathematical work of the teacher in presenting the problem, mediating learner progress, valuing and evaluating their responses, and managing the integration of mathematical content and mathematical processes as foci in the lesson. I present selections from the transcript of the dialogue in the classroom to illuminate these four components of the teachers' mathematical work.

The teacher (Tr), standing in the front of the class, explained what the class had to do.

Tr: I want you to take out a single page quickly. Single page and for the next five minutes no discussion. I want you to think about how would you possibly solve this problem? (pointing to the projected problem: *How many diagonals are there in a 700-sided polygon?*

After seven minutes, the Teacher calls the class' attention. (Learners are referred to as Lr A, B, etc.)

Tr: Ok! Guys, time's up. Five minutes is over. Who of you thinks they solved the problem? One, two, three, four, five, six.
Lr A: I just divided 700 by 2.
Tr: You just divided 700 by 2. (Coughs).
Lr A: Sir, one of the side's have, like a corner. Yes… [inaudible], because of the diagonals. Therefore two of the sides makes like a corner. So I just divided by two… [Inaudible].
Tr: So you just divide the 700 by 2. And what do you base that on? …
[ ]
Tr: Let's hear somebody else's opinion.
Lr B: Sir what I've done sir is … First 700 is too many sides to draw. So if there is four sides how will I do that sir? Then I figure that the four sides must be divided by two. Four divided by two equals two diagonals. So take 700, divide by two will give you the answer. So that's the answer I got.
Tr: *So you say that, there's too many sides to draw. If I can just hear you clearly; … that 700 sides are too many sides, too big a polygon to draw. Let me get it clear. So you took a smaller polygon of four sides and drew the diagonals in there. So how many diagonals you get?*
Lr B: In a four-sided shape sir, I got two.
Tr: Two. *So you deduced from that one example that you should divide the 700 by two as well? So you only went as far as a 4 sided shape? You didn't test anything else.*
Lr B: Yes, I don't want to confuse myself.
Tr: So you don't want to confuse yourself. *So you're happy with that solution, having tested only one polygon?*
Lr B: [Inaudible response.]
Tr: Ok! You say that you have another solution. [Points to learner D] Let's hear.
[ ]
Lr A: I just think it's right… It makes sense.
Tr: What about you Lr D? You said you agree.
Lr D: *He makes sense… He proved it… He used a square.*
Tr: He used a square? *Are you convinced by using a square that he is right?*
Lr E: But sir, here on my page I also did the same thing. I made a 6-sided shape and saw the same thing. Because a six thing has six corners and has three diagonals.
Lr A: So what about a 5-sided shape, then sir?
Tr: *What about a 5-sided shape? You think it would have 5 corners? How many diagonals?*

I have underlined the various contributions by learners, and italicised the teachers' mediating comments and questions. These highlight the learners' reasoning and the teacher's probing for further mathematical justification.

At this point in the lesson, the teacher realises that some of the learners are confusing terms related to polygons, as well as some of the properties of a general polygon and so deflects from the problem for a while to examine with learners, various definitions (of a polygon, pentagon, a diagonal, etc.). In other words, at this point, the mathematical object in which the problem is embedded comes into focus. It is interesting to note here that at no point was there reflection on the polygons in use in developing responses to the problem. All were regular and concave. A little later in the lesson, another learner offers a third solution strategy. The three different solution representations are summarised in Figure 5, illustrating the varying orientations students adopted as they attempted to work towards the solution for a 700-sided polygon.

**Three different representations and reasoning**

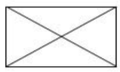
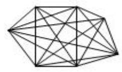
| ***Learner A*** | ***Learner B*** | ***Learner C*** |
|---|---|---|
| 700-sided polygon<br>700 / 2 = 350 diagonals | 4-sided polygon<br>4 / 2 = 2 diagonals | 7-sided polygon<br>14 diagonals<br>14 x 100 = 1400 diagonals |
| Representation:<br>Verbal description | Representation:<br> | Representation:<br> |
| Reasoning: Because of sides – corners. 700/2 = 350 corners and 175 diagonals | Reasoning: Too big a number therefore use a quadrilateral.<br>4/2 = 2 diagonals therefore 700/2 | Reasoning: 7-sided polygon has 14 diagonals therefore multiply by 100 which equals 1400. |

*Figure 5. Three different representations and reasoning.*

As with Example 1, we see four tasks of teaching demanded of the teacher: task design or adaptation; mediation of learners' productions; valuing and evaluating their different responses; and managing mathematics content and processes opened up by the task.

The representations offered by learners give rise to interesting and challenging mathematical work for the teacher. All responses are mathematically flawed, though the approaches of Learners B and C show attempts at specialising and then generalising (Mason, 2002). While this is an appropriate mathematical practice, the move from the special case to the general case in both responses is problematic, though in different ways. Does the teacher move into discussion about specialising and generalising in mathematics (and if so, how)? Open-ended investigations and problem-solving as described above open up possibilities for this kind of mathematical work in class. Such opportunities were not taken up here. Each was negated empirically, and not elaborated more generally. Should they have been taken up by the teacher, and if so, how?

## Tasks of teaching and their mathematical entailments

In selecting and presenting two different examples from different secondary school classrooms in South Africa, I have highlighted four inter-related tasks of teaching, each of which entails considerable mathematical skill and understanding over and above (or underpinning) the teaching moves that will ensue. The four tasks (two of which are discussed in each of the bulleted sections below) further illustrate categories of professional knowledge developed by Shulman and elaborated by Ball et al. in mathematics.

## Designing, adapting or selecting tasks, and managing processes and objects

In the first example, the process of mathematical reasoning was in focus, as was the triangle and its angle properties. I will call this an *object-and-process-focused* task. Angle properties of triangles are the focus of reasoning activity. Learners engage with and consolidate knowledge of these properties through reasoning activity, and vice versa. Here the integration of learning content and process appears to keep them both in focus, and thus provides opportunities for learning both. Example 2 is also focused on mathematical reasoning. It is a *process-focused* task, having been adapted (what I would refer to as recontexualised) from an investigation and re-framed as a problem with a solution. The mathematical object of the activity, the polygon, is backgrounded. At a few points in the lessons, it comes into focus, when understanding polygons and their properties is required for learners to make progress with the problem: some learners make assumptions about what counts as a diagonal, perhaps a function of assuming regularity (and so finding three diagonals in a hexagon); some generalise from one specific case (a four-sided figure); while others over-generalise multiplicative processes from number, to polygon properties.

The intricate relationship between mathematical objects and processes has been an area of extensive empirical research in the field of mathematics education. It appears from studying two examples of teaching that selecting, adapting or designing tasks to optimise teaching and learning entails an understanding of mathematical objects and processes and how these interact within different kinds of tasks. The teaching of mathematical content and mathematical processes is very much in focus today. Reform curricula in many countries promote the appreciation of various mathematical objects, their properties and structure, conventions (how these are used and operated on in mathematical practice), as well what counts as a mathematical argument, and the mathematical processes that support such. In Example 1, we see opportunity for developing reasoning skills, and understanding of proof at the same time as consolidating knowledge about triangles. In Example 2, it is not apparent whether and how either proof or reasoning will flourish through this example and its mediation. The relevance of the mathematical object in use is unclear. Thus the question: *Do we need a mathematics for teaching curriculum that includes task interpretation, analysis and design with specific attention to intended mathematical objects and processes and their interaction?*

In other words, should a mathematics for teaching curriculum include attention to the mediation of mathematical content and processes as these unfold in and through engagement with varying tasks? If so, is this to be part of the mathematics curriculum, or part of the teaching curriculum? And hidden in this last question is a question of who teaches these components of the curriculum in teacher education? What competences and expertise would best support this teaching?

## Valuing and evaluating diverse learner productions

Diverse learner productions are particularly evident in Examples 1 and 2, given their more open or extended nature. Thus, in each example, the teacher dealt with responses from learners that they predicted, and then those that were unexpected. In Example 1, the teacher needed to consider the mathematical validity of Joe's argument for the impossibility of a triangle with two obtuse angles, and then how to encourage him to think about this himself, and convince others in the class. Similarly, we can ask in Example 2: what might be the most productive question to ask Learner C and so challenge the reasoning that, since 700 can be factored into $7 \times 100$, finding the diagonals in a 7-sided figure is the route to the solution to a 700-sided figure? Such questioning in teaching needs to be mathematically informed.

Together these examples illuminate how teachers need to exercise mathematical judgement as they engage with what learners do (or do not do). This is particularly so if teachers are building a pedagogical approach and classroom environment that encourages mathematical practices where error, and partial meanings are understood as fundamental to learning mathematics. In earlier work I referred to this as a teaching dilemma, where managing both the valuing of learner participation and evaluation of the mathematical worth of their responses was important (Adler, 2001); and illuminated the equity concerns if and when evaluation of diverse responses—i.e., judgements as to which are mathematically more robust or worthwhile—are left implicit.

So, a further question needs to be asked of the curriculum in mathematics teacher education, and the notion of mathematics for teaching. Learner errors and misconceptions in mathematics are probably the most developed research areas in mathematics education. We know a great deal about persistent errors and misconceptions that are apparent in learners' mathematical productions across contexts. These provide crucial insight into the diverse responses that can be anticipated from learners. Yet, as Stacey (2004) argues, the development of this research into contents for teacher education has been slow. We have shown elsewhere that the importance of learner mathematical thinking in mathematics teacher education *is* evident in varying programs in South Africa (see Davis, Adler & Parker, 2007; Adler, forthcoming; Parker, 2009). Yet there are significant differences in the ways this is included in such programs, and so with potential effects on who is offered what in their teacher education. *How should a mathematics for teaching curriculum then include such content?*

## Mathematics for teaching matters

I have argued that mathematics for teaching matters for teaching and also for opportunities to learn mathematics. I have suggested that what matters are task design and mediation, as well as attention to mathematical content, objects and processes within these. I have played on the word "matters" by suggesting firstly that these are the "matter" or the content of mathematics for teaching; and at the same time that they matter (have significance) in and for teacher education. Secondly, I have suggested that there are equity issues at stake.

I now return to the context of teacher education in South Africa where various innovative teacher education programs are grappling with a curriculum for mathematics teachers that appreciates the complexity of professional knowledge for teaching and its critical content or subject basis. I will focus here on what we have observed as objects of attention (and so meanings) shift from classrooms to teacher education and back again, observations that support the argument in this paper, that we need to embrace our deeper understanding of the complexities of teaching and so our task in teacher education.

In more activity-based, participative or discursively rich classroom mathematics practice, there is increased attention to mathematical processes as critical to developing mathematical proficiency and inducting learners into a breadth of mathematical practices. The examples in this paper illustrate how mathematical processes are always related to or based on some mathematical object. If the latter is not well understood, in the first instance by the teacher, in ways that enable her to notice when it goes out of focus or is completely missed by students, then their reasoning is likely to be flawed or mathematically empty. This phenomenon is apparent in classrooms in South Africa, and more so in historically disadvantaged settings, thus perpetuating rather then attacking inequality. Mathematical objects and processes and their interaction are the central “matter” of mathematics for teaching. The shift in new curricula to mathematical processes creates conditions for diminished attention to mathematical objects. Attention to *objects and processes* need to be embraced *in the context of teaching* if access to mathematics is to be possible for all learners.

Herein lies considerable challenge. In each of the two examples in this paper, a mathematical object was embedded in a task that worked varyingly to support mathematical reasoning processes. What the teacher in each case faced was different learner productions as responses to the task. These become the focus of the teachers' work, requiring integrated and professional based knowledge of mathematics, teaching tasks and learner thinking. So what then, is or comes into focus in teacher education, and not only into teacher education, but into school curricula? What we have observed (and I have seen elements of this in elementary mathematics teacher education in the UK), is that learner thinking and the diversity of their responses become the focus, with the mathematical objects and tasks that give rise to these, out of focus. What one might see in the case of the triangle properties is a task that requires learners to produce three different arguments for why a triangle cannot have two obtuse angles. And there is a subtle but impacting shift of attention: from how to mediate diverse responses, to multiple answers or solutions being the required competence in learners; from teachers' learning to appreciate diverse learner productions and their relative mathematical worth, and more generally, multiple representations, and how to enable learners to move flexibly between these, to these being the actual content of teaching. Simply, there are curricular texts that now require learners to produce multiple solutions to a problem. I leave this somewhat provocative assertion for discussion and further debate.

In conclusion, there is an assumption at work throughout this paper that teacher education is crucial to quality teaching. In South Africa, all pre-service and formal in-service teacher education has become the responsibility of universities. Tensions between theory and practice abound. I hope in this paper to have provided examples that illuminate the mathematical work of teaching, and through these opened up challenges for mathematics teacher education. Mathematics for teaching, and its place in mathematics teacher education, particularly in less resourced contexts, matters profoundly.

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