Wits School of Education & Gauteng Department of Education

Maths 4 Teachers

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In the last Newsletter we discussed some important issues about the equal sign. It is not only a '*do something*' sign, as many learners understand it, but also a symbol representing a relationship between the numbers on either side of the equal sign. While we focused on issues around the equal sign, we had to focus less on other important mathematical knowledge, such as considering the distributive law, or subtracting correctly. We know that when we work with learners, we often observe learners' misconceptions at multiple levels. In the last Newsletter, for example, we saw that the concept of using opposite operations learned as a rule may result in many errors. At the same time, incorrect use of the distributive law and lack of understanding of working with complex fractions make working with the equal sign even more complicated. Therefore, teachers have to work with misconceptions at multiple levels every day.

In the previous Newsletter, we noted that learners often did not perform the subtraction operation correctly. This Newsletter addresses misconceptions concerning subtraction, as the same errors were found to repeat themselves throughout grades 3 to 7. The nature of misconceptions is such that errors are repeated consistently by learners because of an underlying misconception. This is the case in the subtraction items presented in the next section.

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At the DIPIP launch in November 2008: Left to right, Prof Mary Metcalf, Prof Tawana Kupe Prof Karin Brodie, Prof Yael Shalem (Wits), Reena Ramparsad, Prem Govender (GDE)

A group of teachers, subject facilitators and Wits team leaders at the DIPIP launch

The assessment standards (ASs) from the NCS dealing with operations are written below. We will focus only on subtraction in this Newsletter, and have therefore not re-written the entire AS – only the part that deals with subtraction

Grade 3:
Can perform calculations, using appropriate symbols, to solve problems involving:

- Addition and subtraction of whole numbers with **at least 3-digit** numbers

Grade 4:
Estimates and calculates by selecting and using operations appropriate to solving problems that involve

- Addition and subtraction of whole numbers with **at least 4 digits**

Grade 5:
Estimates and calculates by selecting and using operations appropriate to solving problems that involve

- Addition and subtraction of whole numbers with **at least 5 digits**
- Addition and subtraction of common fractions with the same denominator and whole numbers with common fractions (mixed numbers)

Grade 6:
Estimates and calculates by selecting and using operations appropriate to solving problems that involve

- Addition and subtraction of whole numbers
- Addition and subtraction of common fractions with denominators that are multiples of each other and whole numbers with common fractions (mixed numbers)

Grade 7:
Estimates and calculates by selecting and using operations appropriate to solving problems that involve

- Multiple operations with integers
- Addition, subtraction and multiplication of common fractions
- Addition, subtraction and multiplication of positive decimals to at least 2 decimal places

As you read the contents of the assessment standards, and also look at the test items on page 3, you will note that the grades 3, 4, 6 and 7 test items are all easier than the minimum level specified by the assessment standard for the grade. Table 1 below shows that the best achievement on the subtraction items was obtained by the grade 7 learners, with a 71% achievement rate. The achievement in other grades was lower, even though the items can be considered easy for those grades.

Table 1. Percentage achievement of learners in each grade for the subtraction items in the test papers.

Percentage of learners who achieved the correct answer			
Grade 3	Grade 4	Grade 6	Grade 7
34%	59%	34%	71%

Edward, Louise and Ari planning grade 8 learning materials

MISCONCEPTIONS CONCERNING SUBTRACTION

We focus on four main misconceptions identified by the DIPIP teachers when they analysed these results. The fifth was addressed in the previous newsletter (volume 2, issue 1):

1. difficulties in mentally counting backwards and forwards;
2. incomplete knowledge of 'number bonds';
3. incorrect use of the vertical subtraction algorithm;
4. mixing up what is subtracted from what (the subtrahend and the minuend). and
5. opposite operations and the relational meaning of the equal sign

The items presented in this issue all show the same misconceptions from grade 3 to grade 7. Therefore we first provide each question, with a brief explanation of what learners need to understand and do in order to obtain the correct answer. After that, under the four main themes listed above, we discuss the different ways in which learners could have obtained the incorrect answers, including possible misconceptions that may have surfaced. Lastly, we provide suggestions for ways to teach so that these misconceptions may be addressed in the classroom.

Item selection and analysis of correct answers

In each item given the correct answer is highlighted.

Grade 3: 'Subtract a 2-digit number from a 2-digit number'

90 – 36 = ?

(A) 66 (B) 64 (C) 56 ***(D) 54***

34% of learners arrived at the correct answer D
23% chose option A
18% chose option B
14% chose option C
9% of learners did not answer the question

About a third of learners did this problem correctly. Almost a quarter of learners who did this problem chose incorrect answer A, and nearly a fifth of learners chose incorrect answer B.

In order to obtain the correct answer of 54, learners may have mentally subtracted 30 from ninety, reaching 60; and subtracted 6 more, so obtaining a final answer of 54. Alternatively, they may have used a written subtraction algorithm – either vertical or horizontal – to obtain the correct answer.

Grade 4: 'Subtract a 1-digit number from a 2-digit number'

71 – 8 = ?

(A)63 *(B)64 (C)73 (D)77*

59% of learners arrived at the correct answer A
17% chose option B
9% chose option C
9% chose option D
3% of learners did not answer the question

About three fifths of the learners finished this problem correctly. The teachers who analysed this question thought it was very easy for a grade 4 child. At a grade 4 level learners should be able to do this mentally, but if they needed to use a pen and paper to write an algorithm to find the answer, this is similar to a grade 2 level in the NCS.

If learners did the problem mentally they could have done it in a few different ways. Firstly, they could have added 8 onto the given solutions and found which gave a result of 71. This method demonstrates their understanding of the reversibility of the '+' and '-' signs. Alternatively, they may have done the subtraction mentally as follows:

71 – 10 + 2 = 63 or

71 – 1 – 7 = 63

Both ways demonstrate good understanding of number concept, and how numbers can be broken down into workable 'chunks' to make such subtraction problems easier to do mentally.

Lastly, they may have counted using their fingers 8 backwards from 71, or they may have written an algorithm.

Angie Koalibane, a DIPIP teacher, with Prof Tawana Kupe, Dean of the Faculty of Humanities at Wits, after Angie made an address at the DIPIP launch

Grade 6: "Subtract a 3-digit number from a 3-digit number'

900 – 358 = ?

(A)542 *(B) 552* *(C) 642* *(D)658*

34% of learners arrived at the correct answer A
11% chose option B
13% chose option C
39% chose option D
1% of learners did not answer the question

Note that about 2-fifths of the grade 6 learners chose incorrect option D. This indicates a serious problem, as the question should have been easily managed by a grade 5 learner. To get the answer correct the learners could have done a mental subtraction, or used a vertical or horizontal subtraction algorithm.

Mentally, they could have broken the subtraction operation into parts; saying,
900 – 300 = 600;
600 – 50 = 550;
550 – 8 = 542.

Grade 7: 'Use subtraction to find a missing number'

What is the missing number?

123 + ? = 131

(A) 8 *(B)* 12 *(C)* 18 *(D)* 254

71% of learners arrived at the correct answer (A)
6% chose option B
9% chose option C
10% chose option D
2% of learners did not answer the question

To do this item correctly learners need a clear understanding that the question requires them to find out how much more 131 is than 123, or how much less 123 is than 131. Either way, an appropriate mathematical way to answer this is by subtracting 123 from 131, to obtain an answer of 8; by understanding that addition is the operational inverse of subtraction.

Learners may count forwards from 123 until they reach 131, or backwards from 131 to 123. A mental record is kept of how many units were counted from 123 to 131, or vice versa, and an answer of 8 is reached. Using any of these methods to find the answer requires understanding of opposite operations linked with the relational meaning of the equal sign, required for the solution of equations in grade 8 onwards.

Discussion of the errors in the grades 3, 4, 6 and 7 items

Four major problems (see page 3) can be linked to errors made in the items, resulting in the learners choosing the incorrect answers. We discuss each major problem and relate it to specific problem(s) in each item.

1. The problem of mentally counting backwards and forwards

This problem is best seen in the grade 4 item, which is also appropriate for grade 3 learners. 17% of grade 4 learners chose option B – the answer of 64, instead of 63. They could have obtained this answer by counting backwards from 71 incorrectly – by beginning at 71. To do this they might say, "71; 70; 69; 68; 67; 66; 65; 64;" ending eight digits later at 64, which they then wrote as the answer. This is commonly done by children at this level when counting forwards and backwards: they tend to include the first number. When they work concretely, if they had 71 items and took away 8 it is more likely that they would get the right answer. However, when they work mentally they struggle to visualise adding on or taking away the first unit as an interval; which means that they do not start at the next number up or down from the first. This problem may be addressed by using counting on a number line. We have provided an example of using a number line to count backwards and forwards in activity sheet 1. You can design other similar examples for yourselves.

2. The problem of using incorrect subtraction pairs or number bonds

In the grade 3 item,

90 – 36 = [?]

the subtraction could have been done mentally. It is possible that the learners did not know how to combine their number bonds to be able to extend them to subtracting larger numbers. In this example, they had to extend their knowledge of the bonds of 9 and 10 to subtracting 30 from 90 (9 – 3), and 6 from 60 (10 – 6). These two answers need to be combined to get an answer of 54. It is possible that both 90 – 30 and 60 - 6 could have been done with ease separately, but when combined, the learners could have said something like the following: "90 – 30 = 60 and 10 – 6 = 4; so the answer is 64 (option B), rather than 54.

Similarly, in the grade 4 item,

71 – 8 = [?]

learners should have used their number bond knowledge that 11 – 8 = 3, but also needed to extend that knowledge to '71 – 8 = 63'. This could have resulted in their answer of 64 (option B), instead of 63. It is important that learners in the lower grades know their number bonds, so that mental addition and subtraction with increasingly larger numbers is easier for learners in later grades.

3. The problem of incorrectly using the vertical subtraction algorithm

In the grade 3 item incorrect option B can be the result of using the written vertical algorithm incorrectly. The problem can be written as:

$$\begin{array}{r} 90 \\ -\underline{36} \end{array}$$

Here, a learner might say, "0 – 6 can't be done, so I must borrow 1". Then the child says, "10 – 6 = 4, and then 9 – 3 = 6, and the answer is 64". Such a mistake may be a slip. Alternatively, it may be a more serious result of learning the subtraction algorithm as a rule. This often results in the learner not understanding why the tens value is reduced by 1 ten after 'borrowing'.

Closely associated with 'borrowing' used in subtraction is an understanding of place value of the numbers in each column. If place value is not understood, the column subtraction does not make sense, and is more likely to be learned as a rule, resulting in frequent incorrect answers. We find a similar problem in the grade 4 item.

In the grade 4 item incorrect use of the vertical subtraction algorithm

$$\begin{array}{r} 71 \\ -\underline{\ 8} \end{array}$$

can result in incorrect option C (73). If option C was chosen, it is likely that a misconception exists concerning place value. The misconception relates to the learners not remembering that they have "borrowed" 10 from the ten's place, and so they did not reduce the tens from 70 to 60.

In the grade 6 item incorrect answers B and C can both be obtained as a result of incorrectly using the subtraction algorithm. For incorrect answer B, the learners possibly forgot that they had already 'borrowed' 10 from the tens column, and the learners forgot to change 10 to 9. Thus they subtracted 5 from 10 in that column. The learners might say or think:

$$\begin{array}{r} {}^{8}9{}^{1}0{}^{1}0 \\ -\underline{358} \\ 552 \end{array}$$

10 – 8 = 2 We borrowed 10 from the middle column and to get ten units, and subtracted 8 to get 2

10 – 5 = 5 We borrowed 100 from 900 and made it 100, and subtracted 50. Then we subtracted 300 from 800

In the hundreds column 8 – 3 = 5 was not forgotten

For option C the learners possibly forgot that they had already 'borrowed' 100 from 900 in the hundreds column. Thus they subtracted the 300 from 900 instead of from 800, obtaining a final answer of 642. Again, this kind of mistake could have been a slip, or it can be a result of learners following a method, without having proper understanding of place value, through learning the rules of subtraction without conceptual understanding.

Mmatladi and Peter mapping items onto the NCS

In the grade 7 item, 123 + ? = 131

incorrect option C may also have been a subtraction 'slip', but could also have been chosen through the existence of a misunderstanding of the column subtraction algorithm. Consider the problem and the comments in the boxes. The learners might think or say:

$$\begin{array}{r} {}^{1}\\ 131 \\ -\underline{123} \\ 18 \end{array}$$

We can't subtract 3 from 1, so we 'borrow' 1 from the 3 in the middle column, and we get '11'. Then 11 – 3 = 8.

Then we took 2 from 3: Therefore, 3 – 2 = 1 and the answer is 18

In the first box above, the learner might say, "We can't subtract 3 from 1, so we 'borrow' 1 from the 3, and we get '11'". The most obvious problem with this is that the '3' is not '3', but '30'. The place value here is 3 tens, which is often left out of the explanation altogether. Correct language describes what is happening in reality: We 'built up' the units column into 1 unit plus 1 ten that we 'borrowed' from the tens column, to make 11 units. If we 'borrowed' a ten from 30, then we 'broke down' 30, remaining with 20. We subtracted 20 from 20, to obtain an answer of zero in the tens column. If we are careful about the language we use when teaching this, we can avoid unnecessary confusion associated with incorrect language usage.

It is important to understand that even if we are very careful to use correct language and plan our lessons carefully to associate place value with the subtraction algorithm, learners can still develop the misconceptions described above. We remind you that misconceptions are not the result of 'bad' teaching, but of learners' sense-making. In the next section we show how an explanation that is correct and necessary in grades 1 – 4 can lead to learners developing a misconception about subtraction, because they need to build new understanding about number sense and subtraction that incorporates new concepts.

4. Mixing up what is subtracted from what

In the grade 3 item incorrect option A (66) could have been obtained if learners mixed up the numbers to be subtracted.

$$\begin{array}{r} 90 \\ -\underline{36} \\ 66 \end{array}$$

In this example, many learners first say '6 – 0 = 6' for the units and then '9 – 3 = 6', for the tens, obtaining the answer of 66.

This is a common problem in the early grades, but can extend all the way through primary schooling.

The misconception originates in the early grades, where learners first understand subtraction in terms of taking away physical objects – where it is not possible to take a bigger from a smaller number. This is perfectly valid as an early understanding of subtraction, but when we start using the vertical subtraction algorithm this understanding needs to be re-worked by the learner into a more complex understanding. The concept of borrowing is introduced to make the top number bigger than the bottom number, so that subtraction may be done. If the learners cannot make sense of the borrowing concept they hold onto their initial understanding that a smaller number must be subtracted from a larger one, and hence mix up the top and bottom numbers in the column subtraction to do this operation.

In the grade 4 item we see the same problem. If the learners wrote an algorithm to solve this subtraction problem, they may have done the following: for 71 – 8, we cannot say '1 – 8' and therefore say '8 – 1', which gives an answer of 7. Thereafter, they go on to say 7 – 0 = 7, and obtain an answer of 77, or option D. Learners who chose this option have not looked critically at their answer and asked themselves why the answer is bigger than the original number, when they have just done a subtraction operation of a smaller positive number from a bigger positive number.

In order to get the incorrect answer D in the grade 6 item, learners could have mixed the subtrahend and minuend, as shown:

$$\begin{array}{r} 900 \\ -\underline{358} \\ 658 \end{array}$$

9 – 3 = 6

5 – 0 = 5

8 – 0 = 8

Some of the DIPIP teachers gave this question to their learners so that they could have an idea of why learners might be choosing option D so frequently. They reported that some learners arranged the numbers in columns with the minuend (358) above and the subtrahend (900) underneath, but still subtracted 3 from 9 in the hundreds column. This shows that some learners are uncomfortable with having zeros on top of bigger numbers when they subtract, and want to only subtract smaller numbers from bigger numbers. It also shows that they do not see the number as a whole '900', but are seeing the 9, 0 and 0 as separate numbers, and the columns as separate subtraction problems.

For the grade 7 item incorrect option B could have been obtained in the same way: Learners say, '3 cannot be subtracted from 1' (which is another misconception at grade 7 level, because according to grade 7 learners' new understanding of integers, 1 – 3 = -2). In this solution 1 is subtracted from 3 in the units column, and then 2 is subtracted from 3 in the tens column; leading to an answer of 12.

$$\begin{array}{r} 131 \\ -\underline{123} \\ 12 \end{array}$$

Units column: 3 – 1 = 2
Tens column: 3 – 2 = 1
∴ answer = 12

This shows that even grade 7 learners are showing the common misconception that the smaller number must always be subtracted from the larger.

In summary, the misconception of mixing the subtrahend and the minuend is the result of learners' thinking that they can only subtract a smaller number from a bigger number. We have already suggested that this is the result of an earlier valid understanding that has served the learners well in other earlier situations.

In early primary school learning about the subtraction operation as 'subtracting the smaller from the larger number' applies to whole numbers, natural numbers and counting numbers. Because this understanding was gained through the use of concrete objects, it is firmly held. It later becomes a misconception because it is still firmly held, but is no longer an adequate model for subtraction. Learners now have to work with larger numbers and incorporate place value understanding into the methods they use to do subtraction. They also learn about integers, where it makes perfect sense to subtract a larger from a smaller number.

We have seen the same problems being repeated in the grades 3, 4, 6 and 7 items discussed. As learners progress through primary school, those who mix the subtrahend and minuend when subtracting show an increasingly poorly-developed number concept, because they have not adequately incorporated new understanding of place value, vertical algorithms and integers into their pre-existing knowledge.

5. Opposite operations and the relational understanding of the equal sign

The learners who obtained incorrect answer D in the grade 7 item added 123 and 131 to get 254: probably because of the '+' sign in the question. The fact that 10% of learners chose this option is of concern, as it is seemingly obvious that 8 can be the only answer. Learners who have added the two numbers in the number sentence have poor understanding of how the numbers relate to each other through their position in relation to the equal sign. For further discussion on this issue refer to Newsletter volume 2 issues 1A and B, January 2009.

Summary

Considering all the possible errors and their similarities across all the items from grades 3 to 7, we suggest that the misconceptions of (1) inadequate number concept with respect to relationships between numbers, (2) mixing subtrahends and minuends, and (3) confusing issues of place value, are all interlinked.

Our discussion in the next section will cover some general issues around these three misconceptions and offer some suggestions for teaching to address some of these problems. Thereafter we provide activity sheets that may help you to work through some of the problems with your learners.

Nosisa, Kathleen and Dave discussing some test items

HANDLING THE IDENTIFIED ERRORS AND MISCONCEPTIONS

Working with counting numbers and the meaning of subtraction

Working with the four operations, including subtraction, is specifically referred to in two LO1 assessment standards other than those written on page 2. We include them here because they are the concepts that are involved in the misconceptions we discuss in this newsletter. Paraphrased, the first says,

"…[the learner] *performs mental calculations involving addition, subtraction and multiplication*", at various specified levels, up to grade 6; and also includes simple squared and cubed numbers in grade 7.

The second says, from grades 3 – 6: "…*uses a range of techniques to perform written and mental calculations with whole numbers*". This includes building up and breaking down numbers, rounding off, doubling and halving, using a number line, adding, subtracting, multiplying in columns, and long division. Later in grade 7, the learner "…*uses a range of techniques to perform calculations including using the commutative, associative and distributive properties with positive rational numbers and zero*".

The fact that LO1 describes number operations in a few different assessment standards indicates how important it is for learners to be able to work competently with numbers and number operations using mental and written algorithms. The focus of this newsletter is subtraction. The teaching of subtraction usually follows after that of addition, since subtraction reverses the addition process. If learners do not understand basic addition and subtraction concepts early on, they may experience difficulty when they are expected to perform operations on bigger numbers, fractions and decimals in later grades.

When subtraction is introduced in grade 3 it is taught as "take away". Using concrete items learners take some items away from a pile, according to the given numbers. At this level learners develop an understanding that for subtraction to be done they have to take away the smaller number from the bigger number. This understanding serves them well when they work with whole numbers. With large numbers, fractions, decimals, integers and variables this understanding requires a shift.

One way to help learners to deal with the shift is to help them to understand subtraction as counting backwards from a given number. To gain experience in this, you can get them to first count in 1's, then in groups of 2, 3, 5, etc, backwards from a starting number. For example, for 56 – 9 they will count backwards using a number chart, starting at 55, and counting back nine in 1's, till they reach 47. Then they can count backwards in 3's: the first number will be 53, the next 50, and the next 47. You can try another number and count backwards in 2's; and so on. After this activity they need to be able link the counting backwards with the operation of subtraction. They can do this by subtracting a given amount and writing it down using correct symbolic notation for subtraction. For the first example above, they might write

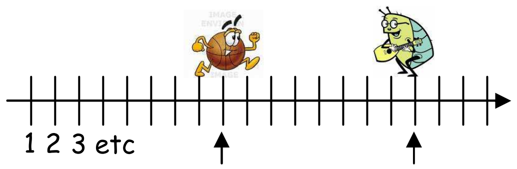
$$56 - 3 = 53$$
$$53 - 3 = 50$$
$$50 - 3 = 47,$$

and note that they took away 9 altogether from 56, to get 47. This is complicated, so this kind of work needs to be done using many examples accompanied by discussion. Learners must be able to explain the link between counting backwards, subtraction, and symbolic notation – not do it through rote learning, drill and practicing a method – otherwise their understanding will be no better than before.

When learners start to count backwards many do not do this correctly. A common error is that they start counting at the first number. For example, for 18 take away 5, they might say, "18, 17, 16, 15, 14", reaching the incorrect answer of 14. You can help them realise how to count backwards as they remove objects, such as counters, from a pile. This will help them to see that the first step when counting backwards takes them *down one* from the starting number. Using number lines and number charts to hop backwards and forwards in 1's, 2's, 3's, 4's, 5's, etc, is also very useful to help learners to visualise how and why they must start at the next number when counting backwards and forwards.

For example, learners could be presented with the following problems:

1. "Fillippe the flea is sitting on number 17. He can see his friend Fenella sitting on number 9. How many hops must he make to also sit on number 9?



2. Calculate 56 – 8 on a number chart, and then on a number line

You can make your own number line out of cardboard. For foundation phase learners and grade 4's we suggest that you make a large one, going from 0 to 100 around the room. Learners in the higher grades should be able to work with portions of the number line, such as from 20 to 80, or 76 to 96, crossing the tens barrier. Note that learners will probably struggle to work with portions of number lines if they have not spent a long time in the earlier grades counting backwards and forwards and subtracting numbers with the aid of their number charts, counters and number lines starting at zero. If they are struggling in the later grades, then take them back to this level of working with subtraction.

In grades 2 and 3 learners need to do a large amount of number work. Many are familiar with number charts, and we suggest you work with these to begin with. While they are using the number chart to physically hop their fingers or objects backwards, they can say the words "take away". Thus, the concept of "take away" is linked with the concept of counting backwards, which is important at this level. There is an important generalisation that you need to help learners to make, by asking questions about their answers. They need to see the pattern for themselves that the answer after "taking away" or counting backwards is always smaller than the starting number. Counting backwards helps them to extend their understanding to negative numbers in later grades.

Much of the work at this level consists of hopping backwards and forwards on the number chart and verbalising the process. Once they have become familiar with the process and verbalisation, learners can work on writing number sentences to symbolise subtraction. To summarise, the steps are: (1) do it; (2) say it; and then (3) write a number sentence. In example 2 at the top of this page, learners will count backwards 8 from 56 using their number charts. They will say, "fifty-six take away eight is 48". Then they will write "56 – 8 = 48". In this way the symbolic representation is linked with both the process and the concept of "take away".

Once the subtraction concept is more familiar, work with a number line, so that learners can see that they do not only have to use a number chart to do subtraction. Also encourage your learners to do simple subtractions, such as 38 – 10; or 65 – 3 by saying and writing it, but without using the number chart to do the physical calculation. They can get an answer using counting backwards, picturing their number charts, and then checking their answers using their number charts or number lines. This encourages grades 3 and 4 learners to become less reliant on concrete aids and to work more abstractly.

Learners need to know the basic number bonds very well. They will need to make links from number bonds to operations with bigger numbers; and the concepts of bonds and multiples can be understood well through use of the counting activities we have already discussed.

In grades 5 and 6 you can work with portions of number lines, as described on the previous page. Working backwards and forwards across a portion of your choice of the number line will help learners to understand why they have to start at the next number up or down from the first number, if this is a problem for them. They can work with more difficult subtraction problems, such as 47 – 32; 133 – 25; and 374 – 128. Here they would be expected to count backwards in groups. For example, for 47 – 32 they could jump backwards 10 from 47; and then another 10, and another 10; and then 2 from the result:

47 – 10 = 37
37 – 10 = 27
27 – 10 = 17
17 – 2 = 15

Note: we are NOT writing it as 47 – 10 = 37 – 10 = 27 – 10 = 17 – 2 = 15. For an explanation of why this is an incorrect way of writing the subtraction process refer to the previous newsletter (volume 2 issue 1 January 2009) where we talk about correct and incorrect use of the equal sign.

Encourage your learners to write down the subtraction problem using mathematical symbols after they verbally explain how the subtraction was done. This will help them to make the connection between the subtraction operation and symbolic representation of subtraction. In later grades subtraction problems such as 374 – 128 may be written using both vertical and horizontal algorithms; and also calculated using the number line to find an answer.

Place value and the teaching of subtraction algorithms

Learners need to understand place value and be able to speak the language of place value in order to be able to use the vertical subtraction algorithm with understanding. Learners who are taught the "recipe" for the vertical algorithm tend to make errors of calculation. For example, using an explanation like, "line up the columns, subtract column by column and borrow when you need to", results in learner errors; especially when learners try to use the vertical subtraction algorithm to subtract for example, 900 – 358.

When learners do not understand that the digits in the vertical algorithm are part of a whole number, they make sense of the subtraction to be done by saying, "the digit in the bottom number is larger than the digit in the top number, and so the subtraction cannot be done". Learners who believe this are focussing on the digits in the separate columns, rather than on the number as a whole.

Teachers can work with this kind of misconception by helping learners to shift their understanding of why it appears that a bigger number is being subtracted from a smaller number. Explain this error to the learners in terms of place value.

Although there are no tens or units in the number 900, the learners have not thought about the fact that 900 is bigger than 358;

because they are too busy working with the units, tens and hundreds places as separate individual numbers.

$$\begin{array}{r} 900 \\ -\underline{358} \end{array}$$

It is crucial that learners do not lose track of the units, tens, hundreds, etc. in the minuend (the top number) and subtrahend (the bottom number) as parts of a whole number. They must also always be aware of how the parts of the whole numbers relate to each other when one is being subtracted from the other.

Estimation can also help learners to think about numbers as a whole. For example, they can think about:
900 – 300 = 600,
600 – 50 = 550,
and so the answer should be around 550. If the 8 is also taken into account the answer should be just less than 550. Estimation also leads to the horizontal algorithm where the subtraction can be broken down into units, tens and hundreds in a different way as shown:

900 – 300 = 600;
600 – 50 = 550;
550 – 8 = 542
Therefore
900 – 358 = 542

This is one of the possible ways of laying out the subtraction of 900 – 358 horizontally. When the teacher speaks about this she is likely to use the language of place value because the process involves breaking the number into hundreds, tens and units and dealing with them as parts of the whole number.

Using this way of writing the subtraction problem will help learners to understand that a 3-digit number, or larger, is not a collection of separate numbers, but is a single number that is made up of hundreds, tens and units. In this way learners are able to observe both how subtraction problems may be written, and that each number in the subtraction problem is a single number.

In the vertical algorithm the word "borrow" can create confusion. A better mathematical expression for "borrowing" is "breaking down" the number, using the concept of place value.

"Borrowing" means "breaking down" the number in the next higher place in order to release a group of ten times the lower place value. It "builds up" the number in that lower place because the digit in the lower place is "too small" for the required subtraction.

The correct "breaking down" procedure is shown here.

		9	
	8	~~10~~	10
	~~9~~	~~0~~	0
-	3	5	8
	5	4	2

The learner "breaks down" the 9 hundreds to make it 8 hundreds and 10 tens; and puts the 10 tens in the tens place – so building up the tens. She then "breaks down" one of the tens by breaking the 10 tens down into 9 tens and taking 10 units to the units column. Here we build up the units. This procedure enables the subtraction to be completed correctly.

In conclusion, teachers need to use the language of place value when working with the horizontal and vertical algorithms. They must not brush over the steps and simply talk about "borrowing" without explaining how it works. The word "borrow" is not the cause of the problem, but a lack of clarity as to how the "borrowing" works is the problem. If the teacher consistently uses the language of place value to speak about the breaking down and "borrowing" that takes place when she goes over examples of the vertical algorithm, learners should become familiar with the algorithm.

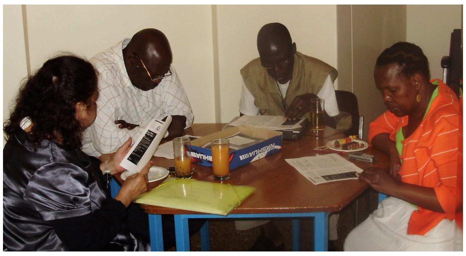
In activity sheet 2 provided below we discuss in detail a way that grade 2 - 6 teachers (and higher grades, if necessary) can help their learners to link the place values of the digits of the numbers found in the vertical algorithm to their places in the whole numbers. This can in turn help the learners to break down numbers, or "borrow", while using the vertical algorithm. It is a little complicated, but it has been found to work very well. We suggest that it can be effective for addressing misconceptions in relation to the use of the vertical subtraction algorithm which are linked to an inadequate application of place value concept.

SOME LEARNER ACTIVITIES

Using the Activity Sheets

The following two activity sheets have been designed to help learners who struggle with understanding place value when doing subtraction, and who develop the misconceptions discussed above.

Left to right: Shadrick, Lisa, Steve, Bennita, Mandla, Tshepo and Setswakai (gr 9) brainstorming lesson plans



Left to right: Sarah, Alfred, Rasheed and Mirriam (gr 6) working with readings about misconceptions

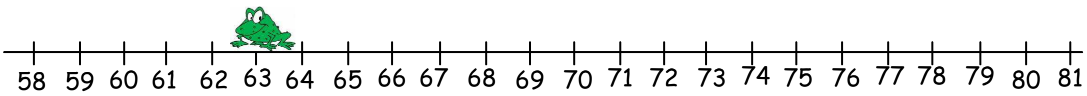
Royston and Euphodia mapping the grade 7 test against the NCS

Activity Sheet 1

Grades 2, 3 and 4

1. a) Count in one's upwards from 59 to 78
   b) Count backwards in one's from 74 to 63
   c) Count in two's upwards from 55 to 85
   d) Count in three's downwards from 81 to 60

2. Ferdinand is sitting on number 63. How many hops must he do so that he is sitting on number71? Write what Ferdinand did in the form of a number sentence.



Write a new number sentence to show what Ferdinand did if he started on 71 and had to hop to 63. How many hops did he need to make to get to 63? What is the difference between 63 and 71?

There is another way to write down what Ferdinand could have done, using exactly the same numbers, but writing a different number sentence. Write the new number sentence down.

3. Nozipho had R54. She wants to buy her Mom a birthday present which costs R78. How much more money does she need? Write down two different number sentences that will describe Nozipho's situation.

4. Tom has 85 marbles in a box. What must he do to have 43 marbles left? Write a number sentence to describe this.

   Kayshree has 43 red counters in a box. What must she do so that she will have 85 counters altogether in her box? Write a number sentence to describe this. Must she always use red counters?

   For the two number sentences you wrote for question 4,
   a) what similarities are there between them?
   b) what are the differences between them?

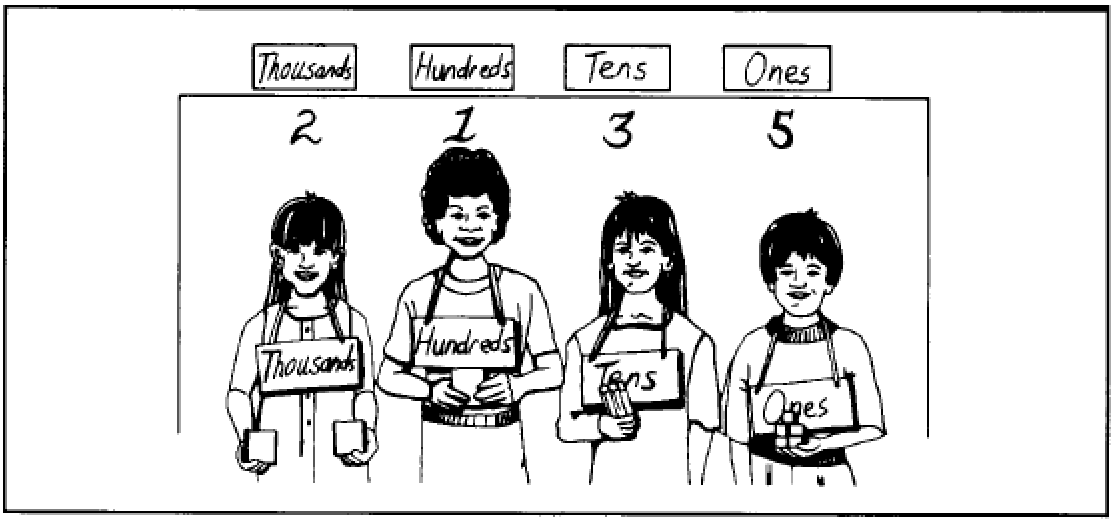
Activity Sheet 2

Place Value Issues with operations

Grades 2 – 6

The ideas for this activity sheet are taken from a journal called *The Arithmetic Teacher* (1990), volume 37 no. 5 pp. 6-9.

We recommend that these activities can be used for children from grades 2 to 6, but that you must select carefully the size of numbers and the kinds of operations to be done with your learners. We have provided ideas and examples below. We suggest that no matter which grade you are teaching, start with the same simple numbers and operations; so that your learners grasp the concepts of place value linked with subtraction. Thereafter you can use more complicated subtraction problems, depending on the grade you are working with.



In the picture the children have been given the instruction to demonstrate the number 2135. In the activity your learners will actually stand at the front of the classroom, doing the same. You can see in the picture that each child is holding the same number of objects as the number in the units, tens, hundreds or thousands he or she is physically representing. Note also that the objects are different. The 'ones' are small blocks; the 'tens' are rods, and so on.

We suggest you start with a small number, such as 26. To represent the number 26, two children will be selected to 'be' the number. One child will be the 'tens' and the other the 'units'. The child who is the tens stands with a 'tens' label around her neck, and has to decide how many tens blocks she should hold. She needs to understand why she should select two tens blocks to hold, and the class should also understand why. The other child will have a 'units' label around his neck, and will have to select the correct number of units blocks to hold. Similarly, both he and the class should understand why he should hold 6 units blocks. The two children would be physically representing the number 26 in front of the class; but the class is involved with deciding whether or not the children at the front have accurately represented the number they were given. Once the process is clear to the children, bigger numbers may be used.

If you have very big classes, you can break up into smaller groups after you have done the exercise with the whole class a few times. This will allow your learners to work more independently, and also allow more learners to participate fully. After a few practice runs you can begin your activity:

1. Write a list of numbers on the board - a big enough list to allow each child to have an opportunity to stand up and represent part of the number written down. Because you would be introducing this activity for the first time in whichever grade you are teaching, we suggest you start with the same small numbers; regardless of the grade you are working with. Then, depending on the grade, you can use increasingly bigger numbers, so that your learners can extend their understanding to the level expected in their grade. Because this activity is so concrete, you can extend your numbers past the requirements of the NCS. Remember that the requirements of the NCS are the minimum required for that grade – so experimentation with bigger numbers is encouraged. You can even ask your learners to make up their own numbers (you should restrict the number of digits) and give them to their classmates as a challenge. They can think of the most difficult numbers possible, within the restrictions you have set, to represent using this activity. In this way, the children are thinking about place value by actually setting challenges, as well as doing what the teacher sets for them. We suggest the lists in the table on the next page as an example.

Work with the numbers given in the table as follows: When a child is given the role to represent the 'tens', say, in the number 4052, s/he must first go to the appropriate box containing the tens objects, select 5 objects to represent the 5 tens in the number, and stand in the correct place next to the other children (to the left of the units child, when facing the class). In this way, an individual child is given the responsibility of selecting the correct place value and positioning him/herself correctly next to the other numbers. The whole class is responsible for making sure that the number represented by the children standing at the front is the same as that written on the board. Once you are satisfied that the children all know what is expected of them you can move on to other activities, depending on what you want to focus on in the NCS, and at which level.

Grade 2	Grade 3	Grade 4	Grade 5	Grade 6
5	5	5	We suggest you follow the same numbers to begin with, as grade 4, but extend your numbers to 100 000's and millions. Note that grades 5 and 6 will be working with fractions and decimal fractions, as well as large whole numbers. Therefore, you need to make sure that they fully understand place value of numbers, as well as the vertical algorithms described below before you try to teach place value of decimal fractions. This may also be a useful activity for teaching place value in decimal fractions: a concept that many learners find difficult to grasp.	
14	14	14		
23	23	23		
46	46	46		
99	99	99		
100	100	100		
107 (zero's are important)	107	107		
143	152	152		
160	199	199		
187	200	200		
199	255	255		
200	299	299		
230	300	300		
279	578	578		
318	624	624		
From here you can experiment, as you see fit, to allow the children to work with bigger numbers	999	999		
	1000	1000		
	1209	1703		
	1583	1999		
	1999	2000		
	2000	5084		
	From here you can experiment, as you see fit, to allow the children to work with much bigger numbers, up to 10 000	6999		
		9999		
		10 000		
		12 000		
		15 045		
		23 460		
		From here you can experiment with much bigger numbers, up to 100 000		

2. A second activity is addition. Here we introduce a 'banker' who is responsible for exchanging objects between the thousands, hundreds, tens and ones. For example, a class can be given the problem '8 + 7' to start with. The child holding 8 'ones' objects is joined by a child holding 7 'ones'. The class concretely sees the problem when there are now too many 'ones', and we need to make a plan. The banker is called in and gives one of the children 1 'ten' object in exchange for 10 'one' objects. That child then stands in the place of the tens, and the other child remains where she is, holding the remainder of the 'ones'; of which there are 5. Thus we concretely see that 8 + 7 makes 1 ten and 5 ones, or 15; and how numbers are broken down and built up in 'carrying'. In the higher grades, we would still work with the banker, but the numbers worked with will be bigger - requiring more exchanges. Start with small numbers, give your class a clear idea of what is happening, and move to working with bigger numbers. This concrete work is valuable because it is so visual. Learners will need to do many numbers to consolidate the idea. While on the topic of addition, you can take this visual concrete way of replacing 10 unit objects with 1 ten object and represent it as a vertical algorithm. For example,

$$\begin{array}{r} 26 \\ +\ 8 \\ \hline \end{array}$$

The banker becomes imaginary and when the learners add 8 and 6, to get 14, they give ten units to the imaginary banker, who exchanges them for 1 ten and puts it by the tens column, with the 2 already there. The learners write down the remaining number of units as '4' in the units column. Thus, the algorithm now looks like this:

$$\begin{array}{r} {}^{1}26 \\ +\ 8 \\ \hline 4 \end{array}$$

The extra 'ten' is added by the learners to the tens they already have, and the result is 3 tens, written in the answer for the tens place:

$$\begin{array}{r} {}^{1}26 \\ +\ 8 \\ \hline 34 \end{array}$$

Thus, you can work with making as smooth as possible the transition between the concrete activity of actually using the learners themselves as placeholders, and the use of the vertical algorithm for addition.

3. Subtraction needs to be planned carefully, but you can work with the banker and the place value objects the same as you did with addition. Before you introduce the banker, first work with numbers that do not require 'borrowing'. For example, start with 8 - 5; move to 16 – 4 or 58 – 13. We suggest that these subtractions be done by asking the units and tens learners standing in the front of the classroom to put back a specified number of the objects they are holding to demonstrate the subtraction being done. The rest of the class can help them to do these actions so that everybody can understand what you are trying to demonstrate. The subtraction of 58 – 13 can present a problem later, so we suggest you work as follows: The subtraction 58 – 13 should be done without using the 'banker' concept, by first subtracting 3 from 58 and then subtracting 10; not the other way around (subtracting 10 first and then 3). Although both ways are equally correct, the banker concept would need to be used in the second method, which complicates the borrowing idea later. So let us continue with the first method.

Two learners are standing at the front of the class: one holding 5 'ten' objects and the other holding 8 'unit' objects. The class must help the 'units' learner to remove 3 units and put them back into the box; leaving her holding 5 units. After that, the 'tens' learner puts 1 ten back into the box, leaving him with 4 tens. Thus the answer, concretely visible, is 45.

When you introduce borrowing start with a simple demonstration of using the banker. For example, for 63 – 7, two learners will stand in the front of the class representing the number 63. The units learner cannot remove 7 from her 3 units; so she has to ask the banker to change 1 ten to 10 units, so that she can have 13 units altogether. Now she is able to subtract 7 from her 13 units, leaving her with 6 units. In the meantime, the banker has taken 1 ten away from the tens learner, leaving him with 5 tens. Thus, the answer, concretely observed by the whole class, is 5 tens and 6 units, or 56.

Note: You as the teacher have a choice to tell your learners how to go about concretely doing the 63 – 7 subtraction using the banker, and how to borrow 1 ten to make 10 units to show how to do the subtraction. Alternatively, and preferably, you can ask your learners to think and discuss their way through deciding for themselves that they need to borrow 1 ten (and not 2 or 3 or more) and exchange it for 10 units, to add to the existing 3 units, in order to subtract 7. To do this you will need to listen carefully to learners' suggestions, ask them carefully-phrased questions to help them think about what they need to do to solve the problem, and *why*.

From here you can move to a subtraction problem, such as 87 – 48, still using the concrete 'banker exchange' idea.

4. The vertical algorithm needs to be introduced once the learners have understood concretely about borrowing from the tens to make enough units to subtract. Work with a simple subtraction problem such as 14 - 8. Children working with a number as small as this should not have problems with mixing what is subtracted from what, and they will easily agree that 14 – 8 = 6. Present the problem as a vertical algorithm:

$$\begin{array}{r} 14 \\ -\ 8 \\ \hline \end{array}$$

Do the subtraction concretely using the banker and the borrowing concept first. Having the banker take 1 'ten' object away from the 'ten' child and give 10 'ones' to the units child will allow the children to visually understand the borrowing process. While doing these activities you can also bring in the concepts of 'break down' and 'build up'. They are 'breaking down' 1 ten into 10 ones and 'building up' the units by including the 10 ones with the units already there. You could then have a discussion at this point about the vertical algorithm and how the borrowing is written down in algorithm to help understand what is really happening. You can help your learners to start to imagine the banker lending a ten, in the form of 10 units, to the units column to make the subtraction possible.

If you are teaching learners who already have the misconception of mixing the top and bottom when subtracting, you can also talk at this point about why saying '8 – 4' will give the wrong answer. The class can discuss what happens when the smaller number (4) is subtracted from the bigger (8), they are mixing the subtrahend and minuend, and getting the wrong answer of 4 instead of 6. You can help your learners to conclude that care needs to be taken that they do not mix what is on top and what is underneath.

Easy subtractions can be done with higher grades to start with, to allow the students to understand 'borrowing', 'breaking down' and 'building up', before they move on to larger numbers with more subtraction complexities. Having your classes do this kind of activity regularly for different operations and for differently-sized numbers will allow your learners to get into the habit of visualising the borrowing process; which will help their understanding. Going too quickly through the process or starting with numbers that are too complex will defeat the objective of rectifying misconceptions around place value and subtraction.

Notes:

- Please let us know if you have tried this activity with your class and tell us how it went. Do you have ideas that will improve the activity? Do you have ways we can use it to teach decimal place value, for example?
- We have focused on subtraction after introducing the activity using addition. You can experiment with multiplication and division as well. Since long multiplication and division have additional conceptual pitfalls, we suggest you try the ideas within this activity between yourselves as a team of teachers before you use it in the classroom.
- An Australian study detailed a case study done in schools, called the "Base Ten Game", that was designed to address the place value problem experienced by teachers in primary schools. It is a large document, but makes for very interesting reading – and it extends the place value concept to decimal fractions. It is worth down-loading and reading for yourself. You can contact us to borrow the article, or it can be obtained in down-loadable form at the URL:

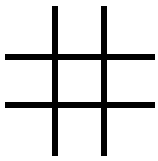
  http://www.dest.gov.au/NR/rdonlyres/5D4B0095-AA36-4ED9-BD50-07A931B22C91/1645/understanding_place_value.pdf

We hope this has been helpful to you for understanding why learners struggle as they do with working with number sentences; and also to give you an idea of some useful tasks to assess where the misconceptions and lack of understanding might be, with respect to subtraction, and try to address problem areas. It does not answer all the questions you might have. If you want to raise some points for discussion please contact us at DIPIP@wits.ac.za

PUZZLE

Try This Problem …

Arrange the numbers 1 through 9 on a tic tac toe board such that the numbers in each row, column, and diagonal add up to 15



Solution to "MURDER THEY WROTE LOGIC PROBLEM" in volume 2 issue 1:

Here's the logic:

The dentist murdered a cousin in Halifax where the motive wasn't revenge or love (clue 3), an inheritance (Brighton, clue 1) or power (solicitor, 5), so blackmail. The revenge killing wasn't in Fishguard or Dunoon (2) so Grantham. The love-motive murder wasn't (4) in Fishguard, so Dunoon, thus the motive in Fishguard was power. The artist's motive was an inheritance (1). The partner (2) was murdered for love by (4) the plumber. The butler killed for revenge. The sister (4) was murdered for revenge. The friend wasn't (5) the victim of a power struggle, so inheritance. The power motive involved the murder of the mother.

Thus:

Artist - friend - inheritance - Brighton

Butler - sister - revenge - Grantham

Dentist - cousin - blackmail - Halifax

Plumber - partner - love - Dunoon

Solicitor - mother - power – Fishguard