Solutions Unit One: Exploring What It Means To 'Do' Mathematics

From the module:
Teaching and Learning Mathematics in Diverse Classrooms

South African Institute for Distance Education

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2017
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Acknowledgements

The South African Institute for Distance Education wishes to thank those below:

For writing and editing the solutions guide to the module:

- Celeste Bortz – writer
- Ingrid Sapire - writer
- Commonwealth of Learning (COL) – for the OER instructional design template
- Tessa Welch – project manager

For permission to adapt the following study guide for the module:

UNISA (2006). *Learning and teaching of Intermediate and Senior Mathematics (ACE ME1-C).* Pretoria: UNISA

For permission to use in Unit One:

- RADMASTE Centre, University of the Witwatersrand (2006). Chapters 1 and 2, *Mathematical Reasoning* (EDUC 263).
- UNISA (2006). Study Units 1 and 2 of *Learning and Teaching of Intermediate and Senior Phase Mathematics*.
- RADMASTE Centre, University of the Witwatersrand (2006). *Number Algebra and Pattern* (EDUC 264).
- Stoker, J. (2001). *Patterns and Functions*. ACE Lecture Notes. RUMEP, Rhodes University, Grahamstown.

Contents

How the solutions for the module are presented

Overview of content

As an introduction to each activity notes about the teaching and learning of the mathematical content in the activity are given. These notes are intended to inform both lecturers using the materials in a teacher education context and teachers who may wish to use the materials in their own classes. The solutions to the activities are all given in full. Diagrams are given to provide visual explanations where necessary.

How the solutions unit is structured

The unit consists of the following:

- General points for discussion relating to the teaching of the mathematical content in the activities.
- Step-by-step mathematical solutions to the activities.
- Annotations to the solutions to assist teachers in their understanding the maths as well as teaching issues relating to the mathematical content represented in the activities.
- Suggestions of links to alternative activities for the teaching of the mathematical content represented in the activities.

How to find alternative content material

The internet gives you access to a large body of mathematical activities, many of which are available for free downloading if you would like to use them in your classroom. There are different ways of searching the web for material, but a very easy way to do this is to use Google. Type in the address http://www.google.co.za/ to get to the Google search page. You can then search for documents by typing in the topic you are thinking of in the space provided. You will be given many titles of articles (and so on) which may be appropriate for you to use. You need to open them in order to check which one actually suits your needs. To open an article you click on the title (on the screen) and you will be taken to the correct web address for that article. You can then check the content and see whether or not it suits your needs. When you do this search you will see that there are many sites which offer worksheets open for use by anyone who would like to use them. You need to check carefully that the material is on the right level for your class and that there are no errors in the text. Anyone can miss a typographical error and web material may not be perfect, but you can easily correct small errors that you find on a worksheet that you download.

You can also use a Google image search, to find images relating to the topic you are thinking of. This usually saves you a lot of time, because you will quickly see which images actually relate to what you are thinking of and which do not. When you click on the image you like (on the screen), this will take you to the full page in which this image is actually found. In this way you can get to the worksheet of your choice. You can then copy and download the material.

Solutions Unit One: Exploring What It Means To 'Do' Mathematics

Activity 4: Thinking about the traditional approach to teaching maths

Most people have their views on what makes a "good maths teacher". If one analyses these views, good maths teachers essentially fall into two groups. The one group teaches algorithmically, to the test, and generally gets "good results". This type of teacher, at the Grade 12 level produces many A's for his/her school and is usually highly praised for doing this. Many learners who have been taught by this group of maths teachers may be very successful at school maths, but their appreciation of the richness of the maths that they are doing is often shallow. And many learners taught in this way never succeed because they do not have the capacity to retain all of the algorithms and methods of selection of algorithms, and maths becomes a frightening chase to "find the right method". There is another group who teach for understanding. They teach so that their classes understand the mathematical concepts and are able to apply them in a variety of contexts, based on their ability to read and understand questions that they encounter.

Outcomes-based education (OBE) has attempted to bring to the fore the need to teach for understanding. This does not mean that OBE invented teaching for understanding. There always have been and there always will be teachers who teach maths in different ways. What has happened with the introduction of OBE is that through the curriculum design, curriculum planners thought that they would be able to change the style of teaching in South Africa. This has not happened in many schools, because although the introduction of OBE certainly has raised huge amounts of discussion on teaching and assessment, in the workplace we still find a mix of teaching styles, on a continuum between those emphasising algorithms and those emphasising conceptual understanding. Every teacher of mathematics needs to think about the way in which he/she teaches maths. The end goal of teaching is a "pass" in the subject, but the route to that pass can be made more or less meaningful by the teacher's guidance and choices in relation to their approach to teaching. This activity raises questions to get you to think about the "traditional" approach to teaching maths. The key things to bear in mind when working through this activity are:

- "Traditional" maths teachers might be seen (especially from a misconstrued understanding of OBE) as those who teach algorithmically – that is they drill the methods. You need to question this stereotypical view.

- There will always be a range of maths teachers with different teaching styles, on a continuum between those emphasising algorithms and those emphasising conceptual understanding. You need to think about your teaching style and be prepared to challenge some of your existing ideas and methods in order to work towards teaching in a way that would be most beneficial to your learners in the long term.

Activity 4

The traditional approach: three simple examples

Three simple 'problems' (A, B, C) are given to you below (at the Intermediate Phase Level).

A. Work through these problems

Problem A:

In servicing a car the attendant used 45ℓ of petrol at R4,25/ℓ and 2 tins of oil at R8,50 each.

What was the total cost for petrol and oil?

Problem B:

The world's record for the high jump in a recent year was 1,87 metres.

On Mars, this jump would be 2½ times as high. How much higher in metres will it be?

Problem C:

John covers ½ of a journey by car, 1/3 of the journey by bicycle, and walks the rest of the way.

- a) What part of the journey does he cover by car and bicycle?
- b) What part of the journey does he walk?

Solutions to Activity 4: The traditional approach: three simple examples

Solution: Problem A

We need to work out the total overall cost, which means that we need to add the total cost of the petrol to the total cost of the oil, using the given information on prices and quantities.

$$\begin{aligned}
&Total\ cost\ of\ petrol = 45 \times 4{,}25 \\
&\qquad = 191{,}25 \\
&Total\ cost\ of\ oil = 8{,}50 \times 2 \\
&\qquad = 17{,}00 \\
&Total\ overall\ cost = 191{,}25 + 17{,}00 \\
&\qquad = 208{,}25 \\
&The\ total\ cost\ for\ petrol\ and\ oil\ is\ R208{,}25.
\end{aligned}$$

We had to multiply the prices with the quantities purchased to work out the subtotals and then add these two subtotals to find our final total.

Solution: Problem B

To find the solution we need to calculate how high the jump would be on Mars and then subtract the world record (on earth).

$$\begin{aligned}
&Height\ of\ jump\ on\ Mars = 1{,}87 \times 2{,}5 \\
&\qquad = 4{,}675 \\
&Difference = 4{,}675 - 1{,}87 \\
&\qquad = 2{,}805 \\
&The\ jump\ on\ Mars\ is\ 2{,}805\,m\ higher\ than\ the\ jump\ on\ earth.
\end{aligned}$$

We had to first multiply to find the height of the jump on Mars. Then subtract the height of the jump on earth from this height to find the difference.

Solution: Problem C

To find the solution to part a) we need to add the given parts of the journey, using our knowledge of addition of fractions. You need to make sure that they know how to do this, and check that they set out their working clearly.

$$\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}$$

$$So\ he\ does\ \frac{5}{6}\ of\ the\ journey\ by\ car\ and\ on\ his\ bicycle.$$

To find the solution to part b) we need to work out what part of the journey is left. This must be the part that he walked.

$$Done\ \frac{5}{6}\ of\ the\ journey\ so\ \frac{6}{6} - \frac{5}{6} = \frac{1}{6}\ must\ be\ done\ walking.$$

B. Indicate by means of a tick (✓) in the blocks given below which of the mathematical skills listed come to the fore for learners attempting each of these 'problems' using traditional approaches to unpacking and 'solving' the problem.

Skills acquired by learners	Problem A	Problem B	Problem C
Computational skills	✓	✓	✓
Using correct order of operations			
Formulating expressions as a mathematical model	✓	✓	✓
Estimation to assess reasonableness of answer	(✓)	(✓)	(✓)
Problem-solving thinking skills	✓	✓	✓
Investigatory skills			
Exploration of rules and logical thinking			
Self-discovery			

Now do the following.

1 Indicate whether the three problems (A, B, C) are typical of the stereotypical traditional way of questioning learners in South Africa.

2 Refer to the skills mentioned above and explain your response briefly.

3 What skills should we teach our learners now in order to prepare them to cope with the 21st century?

4 Make recommendations to your colleagues on the skills we should teach in mathematics.

Discussion of Part B:

1. The problems (A, B and C) are typical maths problems that can be used in both a stereotypical traditional OR a more learner centred way with learners in a South African context.

2. All three problems involved reading, interpretation and then a computation involving operations with different kinds of numbers.

3. Problem solving and investigation would be key skills that would enable learners to deal with situations that confront them. Giving learners opportunities to develop their independent thinking will help them to develop these skills.

4. Remember that maths concepts and algorithms form an important part of the overall maths teaching programme in a year, but that opportunities to develop learners' independent thinking must be given.

Suggested links for other alternative activities:

- www.primaryresources.co.uk/maths/pdfs/squarenumbers.DG.pdf
- Activity for Grade 6 on discovering how to add unlike fractions:

  http://www.math.com/school/subject2/lessons/S2U2L4GL.html

- http://www.icteachers.co.uk/children/sats/3d_shapes.htm

Activity 5: Computational errors and misconceptions

The traditional view of a mathematics teacher showing and telling learners what to do, standing at the front of the classroom, chalk in hand, is what so many of us see as mathematics teaching. The OBE approach is that the learning of mathematics as a process, which should be characterised by learners' meaningful development of concepts and generalisations.   This is characterised mainly by teachers challenging, questioning and guiding, with learners doing, discovering and applying.

In the development of individual conceptual understanding, many learners misunderstand the concept. This can happen in any class ("traditional" or OBE), where teachers may be teaching in different ways and using different styles. Teachers need to be on the look-out for learners who develop misunderstandings, which are sometimes called misconceptions. As a result of these misunderstandings or misconceptions, they may make computational errors which they do not realise are errors since they believe their thinking is correct. Activity 5 focuses on some of these errors and misconceptions and ways in which teachers can deal with them.

Activity 5

Computational errors and misconceptions

Learners make what appear to be computational errors as a result of a lack of understanding of how things work. Here are a few examples:

Computational/notational/conceptual errors

i $\frac{18}{85} = \frac{1}{5}$ $\qquad \frac{1\cancel{8}}{\cancel{8}5} = \frac{1}{5}$ [Learner cancels the 8's]

ii $\begin{array}{r} 625 \\ +\ 25 \\ \hline 875 \end{array}$ [The learner does not understand the concept of place value].

iii 0,234 is bigger than 0,85 [Since 234 is bigger than 85]

iv 3 (4×5) = (3×4) (3×5) [Since 3 (4+5) = 3×4 + 3×5]

v 8 ÷ ½ = 4 [Since 8 ÷ 2 = 4 or 8 × ½ = 4]

vi $a \times a = 2a$ [Since $a + a = 2a$]

vii Half of 8 = 3 [Since half of the figure 8 is 3, if you cut an 8 in half vertically with a pair of scissors]

and so on.

Write down from your own experience a few more examples in which learners make computational errors as a result of a lack of understanding (misconceptions) of how procedures or rules actually work. Consult with other teachers of mathematics.

1 Discuss this with your colleagues and explore solutions to these problems.

2 Suggest the teaching strategies an innovative teacher could use to avoid such misconceptions from developing in learners.

Note: We encourage you to discuss this with your colleagues – we can learn a lot from each other.

Solutions to Activity 4:. Computational errors and misconceptions

1.1 Some examples of computational errors:

$$\frac{a}{\frac{1}{2}} = 4$$

$$\Rightarrow a = 4 \times 2$$

The learner gets confused with the concept of multiplying by one half on both sides of the equation versus the concept of dividing by a fraction. This could be computed in the following way:

$$\tfrac{1}{2}\left(\frac{a}{\frac{1}{2}}\right) = 4 \times \tfrac{1}{2}$$

$$\Rightarrow a = 2$$

1.2 $\dfrac{\cancel{x}+2}{\cancel{x}} = 1+2 = 3$

In this case the learner has cancelled across an addition operation when, in fact, this expression cannot be simplified further since there are no common factors in this fraction!

1.3 $3.2^x = 6^x$ The laws of exponents have been misunderstood.

The expression 3.2^x cannot be simplified.

1.4 $\dfrac{\cos 30°}{\cos 60°} = \dfrac{30°}{60°}$ The learner does not understand the concept of the cosine function of an angle and cancels the "cos" which is incorrect.

The correct answer is: $\dfrac{\cos 30°}{\cos 60°} = \dfrac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \dfrac{\sqrt{3}}{2} \times \dfrac{2}{1} = \sqrt{3}$ (there are also other alternative methods)

2. After much discussion with your colleagues, you may come to the conclusion that the building of a concept and then the building of other concepts onto that concept is very important in development of the correct understanding of concepts. For example, if you have not fully developed with your learners, the concept of $\cos 30°$ using different sized right-angled triangles with an acute angle of $30°$, most learners will fail to understand why it is not possible to cancel the cosines in Question 1.4.

3. Perhaps the best way to develop an understanding of concepts and thereby avoid the development of misconceptions, is to get back to concrete basics in the forming of the concept. For example, with the error (iii) in Activity Five, one could spend a lesson with the learners drawing up decimals on an enormous class-made cardboard number-line. The learners could explore the positions of different decimals and so would eventually be able to explain why 0,234 is not bigger than 0,85. The learners could then be asked to make their own number-lines - and then plot given fractions and decimals.
   We need to take note that we need to allow learners to move beyond the concrete - it is essential that once the concrete is grasped, concepts and generalizations are taken to the abstract to develop the higher level of thinking required, depending on the learner's grade.

Suggested links for other alternative activities/reading:

- http://www.mathandchess.citymaker.com/f/Article_-_Math_Tutoring_Tip_-_Common_Computation_Errors_Made_by_High_School_Studetns.pdf

- http://www.homeschoolmath.net/worksheets/examples/missing-addend-column-form.htm
- http://www.homeschoolmath.net/teaching/teaching-fractions.php

Activity 7: The verbs of doing mathematics

As teachers, we have to continuously question ourselves about the process of doing mathematics (mathematising) in our classrooms. We have to ask ourselves if we are stimulating learning, posing thought-provoking questions, involving the learners in actively doing mathematics and discovering rules, encouraging enquiry as well as providing real problem-solving situations. We cannot afford to focus only on repetitive drill because this seldom results in understanding. Only when learners understand concepts by 'doing' mathematics in the classroom, will we have been successful as educators.

The verbs describing what action or behaviour is expected from the learners when doing a classroom activity are very important for us to use in our structuring and planning of lessons.

Activity 7

The verbs of doing mathematics

3. Reflect on the collection of 'action verbs' above. Do these verbs clearly indicate the type of action required of the learner during the process of mathematising?
4. Study the NCS for a grade to which you are currently teaching mathematics.
5. Are the Assessment Standards expressed in terms of the action verbs?
6. Make a list of the verbs that you find in the Assessment Standards.
7. Do these verbs clearly indicate the type of action required of the learner during the process of mathematising?
8. Give an example of a mathematical activity that demonstrates the action involved in each of these verbs.

Solutions to Activity 7: The verbs of doing mathematics

1. The 'action verbs', for example, explore, verify, explain, develop, investigate etc. are very good indicators of the type of action required of the learner for an activity. In a particular situation, specific verbs will be used. For example, if a learner has to *verify* a conjecture, he or she may *formulate* a plan in trying to *explain and justify* why the conjecture works. In this case, other 'action verbs' such as *predict* or *solve* would not be used in this scenario.

2. Let us consider Grade 8 of the NCS.
   Why don't you have a closer look at the Assessment Standards of the

LO's of the grade which you teach. Mark the action verbs using a highlighter. What do you find? The NCS is full of action verbs.

3. The Assessment Standards **are** expressed in terms of the action verbs.

4. The following action verbs are taken from the first few Assessment Standards of Grade 8:

   describes illustrates recognizes classifies represents

   compares solves estimates calculates checks judges

   investigates extends explains justifies constructs interprets

5. The above verbs clearly indicate the type of action required of the learner during the process of mathematising. In fact, we should take the time to read our NCS document thoroughly now and again, because the description of what process is needed for a particular Assessment Standard is detailed and well-structured for us to understand what our learners should **do** in our classroom to attain the desired outcomes.
6. Examples of mathematical activities using some of these verbs:

- *explore* as many patterns in this sequence as you can

- *solve* the quadratic equation

- *investigate* the properties of a function

- *explain* how to subtract one number from another

- *predict* what will happen to the function if it is shifted

Suggested links for other alternative activities:

- http://www.education.com/reference/article/what-does-mean-to-do-mathematics/

- http://www.kecsac.eku.edu/doc/PD/Critical%20Issues%2008/kris%20althauser%20critial%20issues.ppt#256,1,Teaching The Whole Child Math Conceptually Versus Procedurally

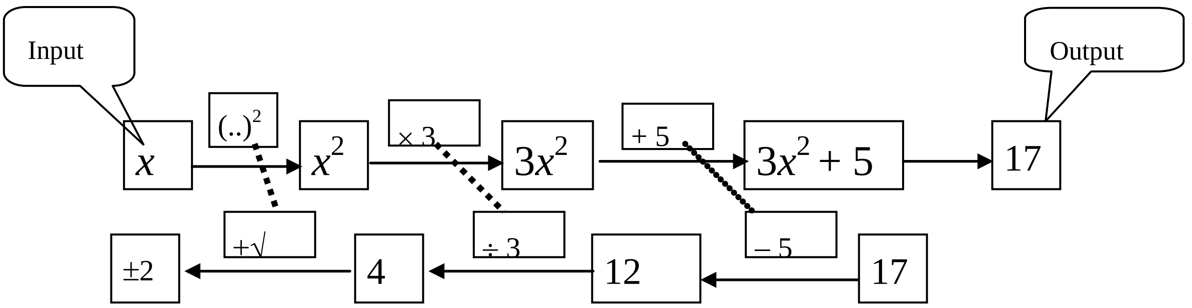
- http://ux1.eiu.edu/~rdanderson/finalexamoutline.pdf

Activity 10: Doing mathematics: informal methods

We as teachers need to ensure that the classroom environment is has a happy atmosphere in which our learners feel secure to have mathematical discussions with their fellow learners or with you, their teacher. We need to set up an atmosphere conducive to exploration, discovery and 'conjecturing'. Learners have to feel respected and therefore free to offer their perceptions without being ridiculed if their perceptions are incorrect. It is also our responsibility to make doing mathematics interesting and fun. There are uncountable informal methods to solve routine problems which make the work that much more exciting for learners since they give the learners the opportunity to work independently, developing their confidence and their mathematical skill..

USING A REVERSE FLOW DIAGRAM TO SOLVE AN EQUATION

Solve for x: $3x^2 + 5=17$



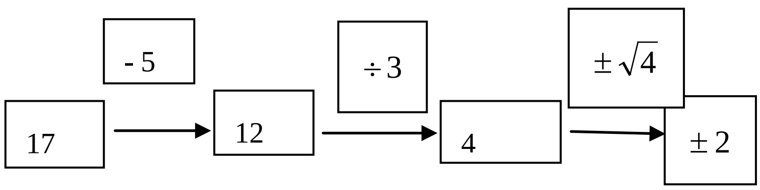
1 Reverse the flow diagram (see the dotted lines to indicate the inverse operations). Start with the output and apply inverse operations. What do you find?

2 Do you agree that the use of informal strategies where learners wrestle towards solutions is never a waste of time? Motivate your response.

3 Compare the above non-routine strategies with the recipe-type routine methods and explain which offer better opportunities for 'doing mathematics' discussions, developing reasons, testing reasons and offering explanations.

4 Have you come across some interesting non-routine methods used by learners in a particular situation? If you have, describe some of these examples. You could also discuss them with your fellow mathematics teachers.

Solutions to Activity 10: Doing mathematics: Informal methods



1. You should find that you get to the solution rather smoothly. The inverse operations starting with the output leads to the input (the x-values) of the original equation.

2. The use of informal strategies is not a waste of time if we as teachers know what our outcomes and time limits are for the lesson(s). Because all learners process their thoughts in their own unique way, informal thinking is very important for developing proper concepts.

3. Recipe-type routine methods originate from the learner-passive method of the teacher-centred approach. This approach works as long as the learner is practicing that section of mathematics. Learners do tend to forget things that have been drilled rather than understood. There is a place for drilling the basics (place value, multiplication tables and number bonds for example). But if the teacher allows learners to develop non-routine problem-solving strategies and allows for greater learner participation, the learner's understanding will be far deeper and possibly more long-lasting.

4. The examples in Activity 10 show some non-routine calculations. Some other interesting developmental or informal methods which can be used as examples are:
    - Grade 10 learners could draw the graph of the parabola by choosing various points and thereby discovering the key points as well as the shape of the parabola. They should then be able to determine a more formal method of sketching the parabola.
    - Throughout the GET and FET phases, number patterns are developed. There are many opportunities for learners to work with informal strategies to develop the formal rules.
    - In Grade 12, once the learner has tried to find the sum of a number of terms in an arithmetic sequence, he or she will want to develop a formula! The learners could be enabled to discover the formula for themselves.

Suggested links for other alternative activities:

- http://math.about.com/od/multiplicatio1/Multiplication_Chart_Worksheets_and_Division_Worksheets.htm
- http://education.jlab.org/smmult/index.html (an online game)
- http://www.mathplayground.com/balloon_invaders.html (an online game)
- http://www.mathsteacher.com.au/year7/ch07_linear/02_solution/equa.htm
- http://www.sosmath.com/algebra/solve/solve0/solve0.html

Activity 11: An environment for doing mathematics

This activity calls for a discussion about the conditions necessary for "doing mathematics" in a classroom. A positive atmosphere in the classroom lends itself to a positive approach to the doing and learning of mathematics. The conversation between the two teachers raises some points about what it takes to facilitate “doing mathematics” in a classroom.

Activity 11

An environment for doing mathematics

Read through the following motivational dialogue between Mr Bright and Mr Spark, two mathematics teachers. Describe the features of a classroom environment which you consider as important for learners to be engaged in doing mathematics.

Mr Bright:	A teacher of ‘doing’ mathematics needs to be enthusiastic, committed and a master of his or her subject.
Mr Spark:	He or she needs to have a personal and easier feel for doing mathematics to create the right environment in the classroom.
Mr Bright:	Teachers should provide activities designed to provide learners with opportunities to engage in the science of pattern and order.
Mr Spark:	Yes, a real opportunity to do some mathematics!
Mr Bright:	We need to develop this technique and discover as much as we can in the process.
Mr Spark:	Let us invite Mr Pattern and perhaps Mr Order and some of the other mathematics teachers.
Mr Bright:	Yes! We would all be actively and meaningfully engaged in doing mathematics and respect and listen to the ideas put forward by the others.
Mr Spark:	We shall challenge each other’s ideas without belittling anyone.

Discussion of Activity 11: An environment for doing mathematics

There are many features of a classroom environment which may be considered to be important for learners to be engaged in doing mathematics. Mr Bright has a keen, determined approach - he simply wants to get on with the doing of mathematics - whereas Mr Spark is very concerned that an atmosphere of warmth, caring and security prevails in the classroom. He seems to want to reiterate that only then can the doing of mathematics take place.

We as teachers need to be enthusiastic and very knowledgeable in our field.

But we also need to have empathy. We need to be able to identify with any difficulties our learners experience.

We need to be well-prepared, providing suitable activities which are exciting and interesting, so that we can engage our learners in the doing of mathematics and thereby discovering and understanding concepts.

Respect between all members of the classroom is one of the most important components for a positive learning atmosphere. Our learners need to feel secure and confident in their environment. They will only be willing to share ideas in the classroom if there is no risk of humiliation.

We can learn from both Mr Bright and Mr Spark's approaches and find a balance in our classroom. In short, we need to be well-prepared, in control, but caring!

Suggested links for other alternative activities:

- http://www.bsrlm.org.uk/IPs/ip17-12/BSRLM-IP-17-12-6.pdf

Activity 13: Start and jump numbers, searching for numbers

Pattern identification and analysis provides a wonderful way of enjoying the doing of mathematics in your classroom. Even those learners with weaker abilities in mathematics can get very involved in pattern finding activities. It is worthwhile having fun and discovering number patterns as in the table below and then extending this to the creation of other number patterns.

Activity 13

Start and jump numbers, searching for numbers (adapted from Van de Walle)

You will need to make a list of numbers that begin with a 'start number' and increase it by a fixed amount which we will call the 'jump number'. First try 3 as the start number and 5 as a jump number. Write the start number first and then 8, 13 and so on 'jumping' by 5 each time until your list extends to about 130.

Your task is to examine this list of numbers and find as many patterns as you possibly can. Share your ideas with the group, and write down every pattern you agree really is a pattern.

Here are some suggestions to guide you:

- Look for alternating patterns
- Look for repeating patterns
- Investigate odd and even numbers
- What is the pattern in the units place?
- What is the pattern in the tens place?
- What happens when you go over 100?
- What happens when you add the digits in the numbers?
- Extend the pattern where you see a gap in the following table.

Which of the following are number patterns (in the columns)?

Complete the table by filling in the empty spaces, and then analyse the numbers in each column:

Separate the consecutive terms		Add terms (in columns 1 and 2)	Subtract terms (in columns 1 and 2)	Multiply terms (in columns 1 and 2)
3	8	11	5 (or -5)	24
13	18	31	5	234
23	28	51	5	644
33	38	71	5	1254
43	48	91	5	2064
53	58	111	5	3074
63	68	131	5	
73	78	151	5	

Solutions to Activity 13: Discussion of some alternative patterns you could find

The above table has been completed below:

Separate the consecutive terms		Add terms (in columns 1 and 2)	Subtract terms (in columns 1 and 2)	Multiply terms (in columns 1 and 2)
3	8	11	5 (or -5)	24
13	18	31	5	234
23	28	51	5	644
33	38	71	5	1254
43	48	91	5	2064
53	58	111	5	3074

63	68	131	5	4284
73	78	151	5	5694
83	88	171	5	7304
93	98	191	5	9114
103	108	211	5	11124
113	118	231	5	13334
123	128	251	5	15744
133	138	271	5	18354

There are so many patterns in the above table. Some patterns are immediately obvious (such as the increase by 10 each time in the first column). Some patterns are not so obvious. For example, in the last column, did you realize that there is a sequence with a 'jump' number of 200? Have fun in your group or on your own in discovering the patterns above. (See how many you and your learners are able to find).

Suggested links for other alternative activities:

- http://www.learner.org/teacherslab/math/patterns/number.html
- http://www.funbrain.com/cracker/index.html (an online game)
- http://www.geocities.com/~harveyh/
- http://www.shodor.org/interactivate/activities/
- http://www.emints.org/ethemes/resources/S00000622.shtml (a whole list of online activities)

Activity 14: Gardening problem

One of the most effective ways to encourage doing mathematics is to present a problem to the learners which is related to their real world. Space and Shape is a Learning Outcome which allows for this in a variety of ways. Models and/or templates can be useful in helping the learners to visualise the problem to be solved. Activity 14 presents a fun and practical way of developing reasoning and problem-solving skills.

Activity 14	**Gardening problem** Jamal had a garden in the shape of a square. Due to the construction of a new road the garden will lose a 3 metre long strip on the south side. Jamal wants to know if he can make up for this difference by adding an extra 3 metres on the east side. 1 Work out the area of the original garden. 2 Cut out a template to represent the area of the garden. Use a scale of 1 cm = 1 m. Mark off the centimetres with a ruler. 3 Cut off a 3 metre strip from the south side of your template. 4 Work out the area of the strip that has been lost. 5 Attach the strip to the east side. 6 What has Jamal not considered if he wants to keep his garden rectangular? 7 To ensure that the garden remains rectangular, what will the area of the strip on the east side need to be for this to be possible? 8 What is the area of the new garden? 9 Does this area differ from the original area at all? If so in what way? 10 Explain the reason for your answer above.

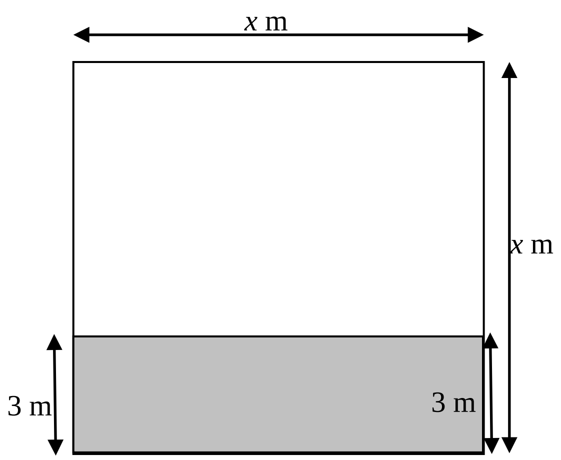
Solutions to Activity 14: Gardening problem

1. We don't know the measurements of the original garden. Let us label the sides of the square x metres.

   Area of original garden: $x \times x = x^2 metres^2$

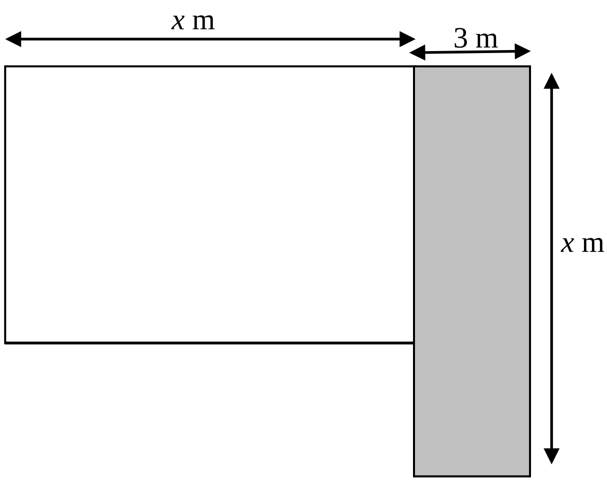
2. When cutting out the template, ensure that your garden has sides which are more than 3m (3 cm on template).

3. A 3 cm strip is cut off from the south side of the template:



4. Area of strip lost: $3x$ metres ^{2} (using the scale)

5. The strip is attached to the east side of the template:

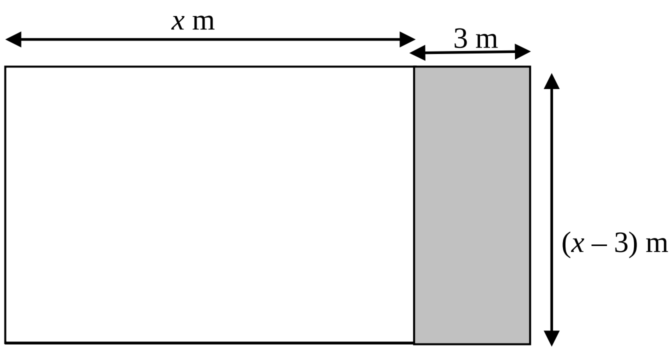


6. If Jamal would like to keep his garden rectangular, he has not considered that the dimensions of the strip which was cut off and moved do not fit in with the left-over dimensions!

7. To ensure that the garden remains rectangular, the strip on the east side will have to be cut down to match the left-over portion.

   Area of strip on east side: $3(x-3) = 3x - 9$ metres^2

8.

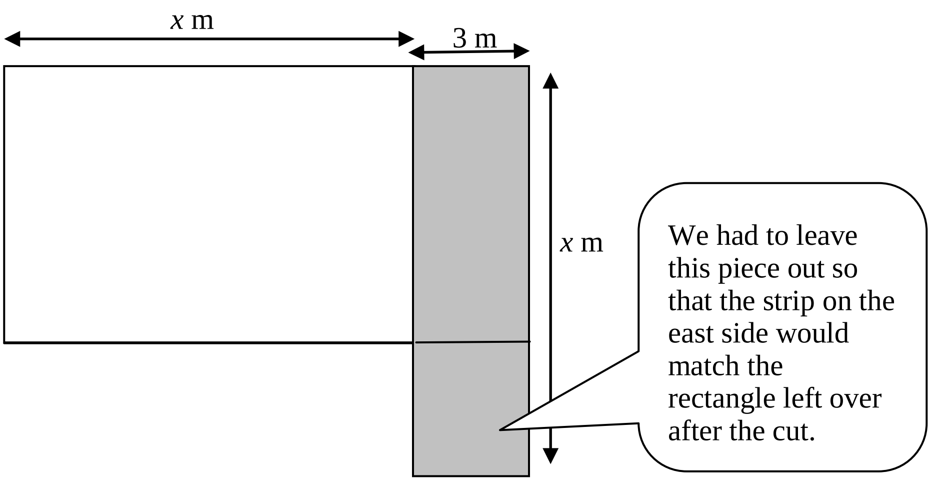


   Area of new garden: $(x+3)(x-3) = (x^2 - 9)metres^2$

9. Original Area - New Area:

$$x^2 - (x^2 - 9)$$
$$= x^2 - x^2 + 9$$
$$= 9m^2$$

10. The new area is $(x^2 - 9)metres^2$ because we had to leave off a square with an area of $3m \times 3m = 9m^2$ to ensure that the garden remained rectangular.



Problems like this are challenging and a practical way to develop reasoning and problem solving skills.

Suggested links for other alternative activities:

- http://nrich.maths.org/public/
- http://nrich.maths.org/public/monthindex.php?mm=1
- http://www.ebook.com/eBooks/Education/Amusements_in_Mathematics (a free ebook)
- http://www.enchantedlearning.com/math/geometry/shapes/
- http://www.apples4theteacher.com/square.html (ineractive on line game)
- http://www.artsconnected.org/toolkit/create_shape_geometric.cfm (on line activity)

Activity 15: Two machines, one job

This is an activity which is commonly posed as a real life problem at the Grade 11 stage. It involves rates of work, since we are working out the amount of work done as time passes.

Activity 15

Two machines, one job (adapted from Van de Walle)

Ron's Recycle Shop was started when Ron bought a used paper-shredding machine. Business was good, so Ron bought a new shredding machine. The old machine could shred a truckload of paper in 4 hours. The new machine could shred the same truckload in only 2 hours. How long will it take to shred a truckload of paper if Ron runs both shredders at the same time?

Make a serious attempt to figure out a solution. (You could use drawings or counters, coins and so on). If you get stuck consider:

- Are you overlooking any assumptions made in the problem?
- Do the machines run at the same time?
- Do they run as fast when working together as when they work alone?
- Does it work to find the average here? Explain your answer.
- Does it work to use ratio and proportion here? Explain your answer.

Solutions to Activity 15: Two machines, one job

The common suggestion as a solution to this problem is to add 4 hours and 2 hours to give 6 hours as the answer. We need to ask the crucial question: "Does it take the same time (that is 6 hours) for the truckload of paper to be shredded when the machines work together or will it take less time?"

Once we realize that both machines working together will take less time than even the quicker machine, then we are ready to start formalizing our solution.

If we consider what happens in 1 hour, the solution to our problem may begin to take shape.

So, the old machine shreds 1 truckload of paper in 4 hours.

Therefore **in 1 hour, the machine shreds ¼ of the truckload**.

The new machine shreds 1 truckload of paper in 2 hours.

Therefore **in 1 hour, the machine shreds ½ of the truckload.**

Together they shred 1 truckload of paper in x hours.

Therefore **in 1 hour, they shred** $\frac{1}{x}$ **of the truckload.**

We can conclude that **IN ONE HOUR:**

$$\frac{1}{4}+\frac{1}{2}=\frac{1}{x}$$

$$LCD:4x$$

$$x+2x=4$$

$$3x=4$$

$$\therefore x=\frac{4}{3}=1\frac{1}{3}$$

Together they take 1 hour and 20 minutes to shred the truckload.
We can't use the average time taken, in this case, because each machine works at a different speed and we don't know these speeds.
We used ratios and proportion for working out what happens in one hour.

Suggested links for other alternative activities:

- http://www.artsconnected.org/toolkit/create_shape_geometric.cfm
- http://www.mathleague.com/help/ratio/ratio.htm

Activity 16: Interpreting bar graphs

Interpreting bar graphs in the DATA HANDLING Learning Outcome, is a very important skill which is used by most of us in everyday life. Bar graphs appear in all forms of the media, such as newspaper and magazines. How we interpret them is key to understanding the message conveyed by the graph. Learner's ability to understand graphs they see in the media will also help them not to be fooled by data which has been poorly represented or even misrepresented. They will learn to be on the look-out for misleading data.

There are many bar graphs available in our daily newspapers. We should cut these out and ask our learners to interpret them. Even better would be to ask the learners to look for this information at home in magazines or in newspapers and then bring it to school with them. They can then have the opportunity to share their findings and interpretations with fellow learners!

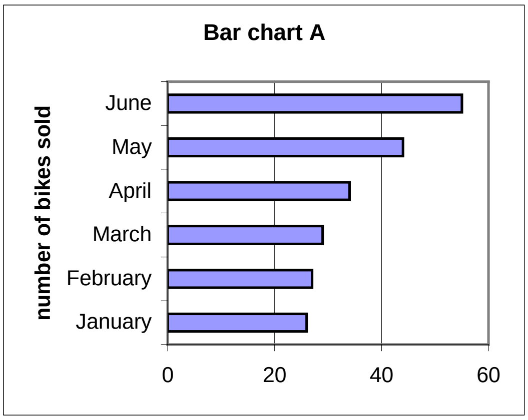
Activity 16

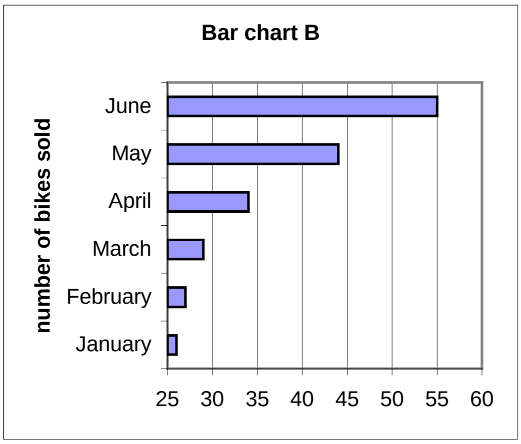
Interpreting bar graphs

Jairos sells bicycles. He is thinking about expanding his business and needs to borrow money from the bank. He wants to show the bank manager that his sales are growing fast. These are his sales for the last 6 months:

Month	Jan	Feb	Mar	Apr	May	June
Number of bikes sold	26	27	29	34	44	55

Jairos draws two graphs as shown below:





1 In what way are the graphs different?

2 Do you think the graphs tell the same story? Why / why not?

3 Which chart do you think Jairos should show to the bank manager? Why?

Solutions to Activity 11: Interpreting bar graphs

1. Bar chart A and Bar chart B are different because the horizontal axis is numbered using different spacing. Bar chart A starts at 0, B starts at 25.

2. The graphs tell the same story. It appears that very little sales were made in January in B. This is due to the horizontal axis starting at 25.

3. Jairos should show the bank manager Bar Chart A because, at a Glance, the sales **look** better. The number of bikes sold are depicted Clearly in this first chart.

Suggested links for other alternative activities:

- http://ims.ode.state.oh.us/ODE/IMS/Lessons/Content/CMA_LP_S05_BA_L10_I06_01.pdf (a full lesson plan with all the necessary data sheets)

- http://mathcentral.uregina.ca/RR/database/RR.09.99/sawatzky1/taskcards-data.pdf

Activity 18: Patterns

We have already spoken about the importance and potential of pattern work when we discussed the solutions to Activity 7 of this unit. The next activity contains many different problems, which are provided for you to use to develop your own pattern recognition skills. You can also use them with your classes, once you have decided that the level of difficulty is appropriate for the grade you are teaching. You may adapt these problems and make them more or less difficult by providing more (or less) scaffolding of the steps required to solve the problem.

Activity 18

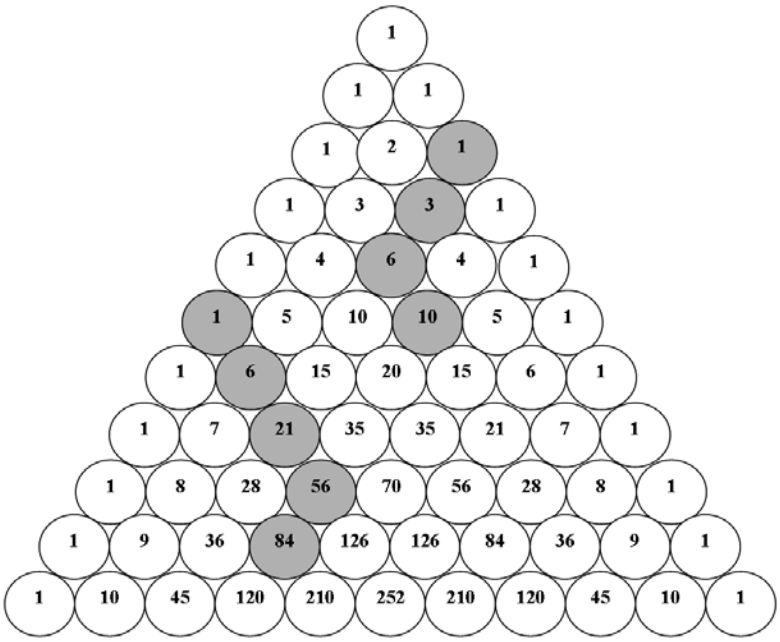
Patterns

1 Pascal's Triangle and the Leg-Foot Pattern

Pascal's Triangle is a fascinating display of numbers, in which many patterns are embedded. In Western writings the Pascal Triangle was named after Blaise Pascal, who was a famous French mathematician and philosopher. Chinese mathematicians knew about Pascal's Triangle long before Pascal was born, so it is also called a Chinese Triangle. It was documented in Chinese writings 300 years before Pascal was born.

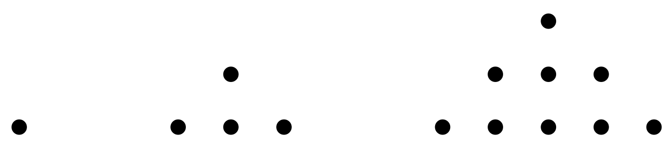
In the triangle below, some "leg-foot "patterns have been shaded. Can you shade more of the patterns that make Leg-foot in this Pascal's Triangle? First look at the two examples that have been done for you. Then shade some more leg-foot patterns in the triangle.

- Now look at the numbers in the patterns that you have shaded.
- Can you see a relationship between the numbers in the shaded in the leg and the foot of the leg-foot patterns that you have shaded? Describe this relationship in words.
- Write a numeric rule for the relationship you have identified.



2 Study the following patterns and then extend them by drawing in the next two stages.

1 **2** **3**



1 **2** **3**

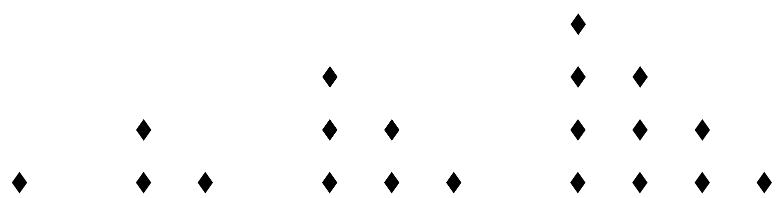
- Draw a table to display the information above:
- What type of numbers are these?
- Use the following to show different representations of square numbers

  i Square grid

  ii Isometric dotty paper

  iii Square dotty paper

3 The following pattern will help Sipho to calculate the number of blocks he will need to build the stairs of his house.



- Help him find the number of blocks he will use for 15 steps.
- Extend this pattern by drawing the next 2 stages of these steps.
- Enter your data in the following table:

Step	1	2	3	4	5	8	15	20	100
Number of blocks	1	3	6						

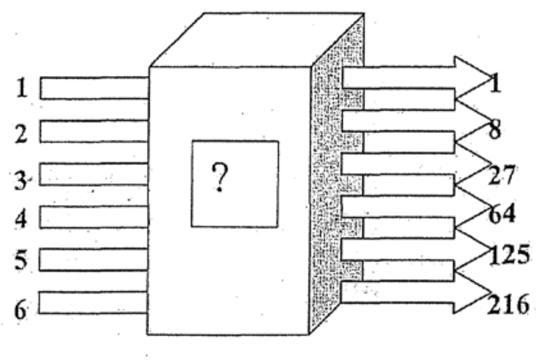
- Find and explain a rule that generates the above pattern.
- What type of numbers are these?
- Use the following to show different representations of triangular numbers:

  iv Square grid

  v Isometric dotty paper

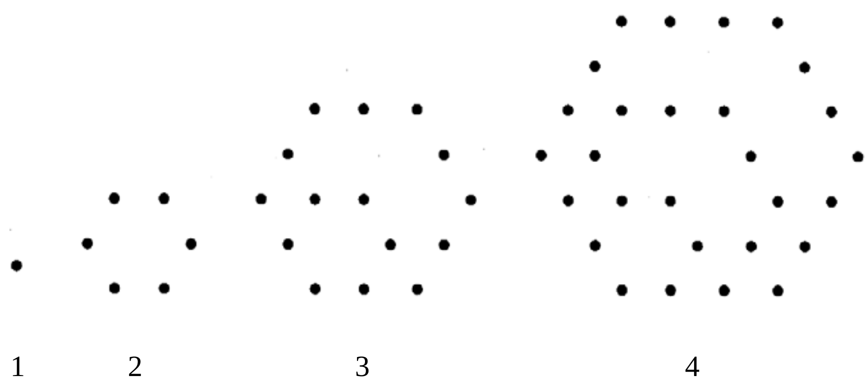
  vi Square dotty paper

4 The following function machine creates a number pattern.



- Investigate the rule that it uses. Write up your findings.
- What type of numbers are these?

5 Study the following number pattern and then complete the table that follows:



Stage	1	2	3	4	5	8	15	20	100
Number of dots	1	6	15	28					

- Investigate a general rule that generates the above pattern.
- What type of numbers are these?

Solutions to Activity 18: Patterns

1. The relationship is such that the number in the foot is the sum of all numbers in its leg.

2.

STAGE	1	2	3	4	5
Squares	1	4	9	16	25
Dots	1	4	9	16	25

The above patterns represent square numbers.

You should also get your learners to show the different representations of square numbers on square grids, isometric dotty paper, square dotty paper if you have the necessary paper available. They can colour the blocks or join the dots to draw the developing patterns themselves. This is an active way of "doing mathematics".

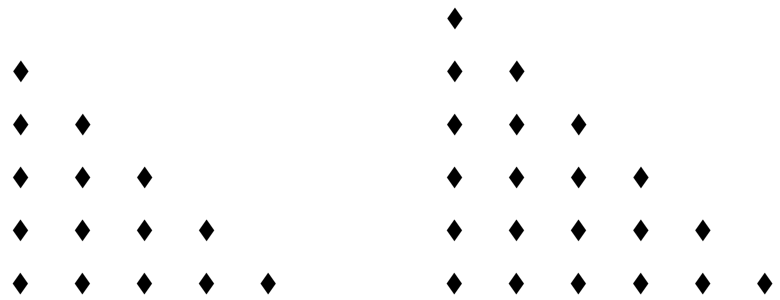
3. Sipho's pattern is based on the following:1; 3; 6; 10; 15; 21; 28; 36; 45; 55; 66; 78; 91; 105; 120; …..

   The first difference between successive terms is: 2; 3; 4; 5; 6; 7; 8; 9; 10; 11; 12; 13; 14; 15; ……

   The second difference between successive terms of the first differences is: **1**

The number of blocks he will use for 15 steps is **120.**

The grid below extends the table and shows the next two stages of this pattern. If you have grid paper available, you should ask your learners to extend the pattern by drawing the next few stages in the pattern.



Here is the completed table of data for Sipho's pattern.

Step	1	2	3	4	5	8	15	20	100
Number of blocks	1	3	6	10	15	21	120	210	5 050

How can you work this out? Every time you make the next stage in the pattern (which is known as the triangular number pattern) you simply add the next natural number to what you have. Numerically, this is how the pattern develops:

1st number: 1

2nd number: 1+2

3rd number: 1+2+3

4th number: 1+2+3+4

5th number: 1+2+3+4+5

And so on…

The number of blocks needed for the 20th and the 100th steps is large! Let's use a rule to generate the pattern and then it will be quite simple to determine the number of blocks needed for the 20th and 100th steps.

So to find the 20th number (using the number pattern demonstrated above) we have to add the first 20 natural numbers. This kind of reasoning to discuss the pattern is appropriate for GET level learner.

Here is an algebraic solution, showing the working required to find a general algebraic rule for the pattern. This is appropriate at the grade 11 or 12 level.

As we mentioned above the second difference is constant, so the general term will be a quadratic expression based on: $T_n = an^2 + bn + c$

$$2a = 2nd\ difference$$

$$2a=1$$

$$\therefore a = \tfrac{1}{2}$$

$$\therefore T_n = \tfrac{1}{2}n^2 + bn + c$$

$T_1 = \tfrac{1}{2}(1)^2 + b(1) + c$

$1 = \tfrac{1}{2} + b + c$

$\tfrac{1}{2} = b + c \quad \ldots(1)$

$\times 2 : 1 = 2b + 2c \quad \ldots(3)$

$\underline{-1 = -2b - c}$

(3)-(2): $\quad 0 = c$

$T_2 = \tfrac{1}{2}(2)^2 + b(2) + c$

$3 = 2 + 2b + c$

$1 = 2b + c \quad \ldots(2)$

Subs. $c = 0$ into (2): $\quad 1 = 2b + 0$

$\therefore \tfrac{1}{2} = b$

$$\therefore T_n = \tfrac{1}{2}n^2 + \tfrac{1}{2}n$$

This is the rule - the general term - which will generate the pattern of **triangular numbers.** Using this formula we can determine T_{20} and T_{100}.

$T_{20} = \tfrac{1}{2}(20)^2 + \tfrac{1}{2}(20) = 210$

$T_{100} = \tfrac{1}{2}(100)^2 + \tfrac{1}{2}(100) = 5050$

4. The rule that the function machine uses is that it cubes the term number. $T_n = n^3$ is the general term which will generate the values. The number pattern is $1^3;2^3;3^3;$......

These are **cubic numbers**.

5.

Stage	1	2	3	4	5	8	15	20	100
Number of dots	1	6	15	28	**45**				

Once again we will present different kinds of reasoning that could be used to solve this pattern problem. The first involves reasoning with the numbers in the sequence, and predicting how to continue the pattern based on this reasoning, but it does not use any algebra. The second, (FET level thinking) provides the algebraic solution for the general term of the sequence.

1st number: 1

2nd number: 1+5

3rd number: 1+5+9

4th number: 1+5+9+13

5th number: 1+5+9+13+17

And so on…

If we look at this, there is a pattern. Every time we add a number which is 4 bigger than the number we added the previous time. So to get the next (6th) number we will have to add 21 on to the 5th number. This is how we can get to the 15th number. It is tedious to generate terms in this way, and at the GET level, analysis such as this would be sufficient, and you would not expect your learner to get to the 100th term unless you wanted them to spend a lot of time on the problem. They may generate a rule without using the advanced algebra below, which you should encourage (but not expect) them to do.

6th number: 1+5+9+13+17+21

7th number: 1+5+9+13+17+21+25

8th number: 1+5+9+13+17+21+25+29

9th number: 1+5+9+13+17+21+25+29+33

10th number: 1+5+9+13+17+21+25+29+33+37

11th number: 1+5+9+13+17+21+25+29+33+37+41

12th number: 1+5+9+13+17+21+25+29+33+37+41+45

13th number: 1+5+9+13+17+21+25+29+33+37+41+45+49

14th number: 1+5+9+13+17+21+25+29+33+37+41+45+49+53

15th number: 1+5+9+13+17+21+25+29+33+37+41+45+49+53+57

Here is how you could find the general term using algebra for an FET group:

To determine the rest of the number of dots in the above stages, it is easier to determine the general rule for this number pattern first.

Terms in the sequence: 1; 6; 15; 28; 45; …..

1st difference: 5; 9; 13; 17; …

2nd difference: 4

Therefore the general term will be a quadratic expression based on $T_n = an^2 + bn + c$ where $2a = 4$ (the 2nd difference), so $a = 2$.
Using the method of simultaneous equations in number 3 above, we will determine the rule (general term) as being $T_n = 2n^2 - n$

Using this formula, we find that the eighth stage requires 120 dots, the fifteenth stage 435 dots, the twentieth stage 780 dots and the one hundredth stage 19900 dots!! (we get this by substituting into the rule)

Suggested links for other alternative activities:

- http://nlvm.usu.edu/EN/NAV/grade_g_3.html (virtual manilpulatives)
- http://www.ccnymathinstitute.net/archive_2006-2007/high/session06/Sixth_Grade_Lesson_Plan.pdf (a full lesson plan)
- http://mathforum.org/workshops/usi/pascal/pascal_numberpatterns.html (using Pascal's triangle)

Unit mathematical content summary

Summary of content covered in this unit:

- Activity 4 involved mathematical concepts relating to problem solving in contexts involving units if measurement.
- Activity 5 involved mathematical concepts relating to the operations and misconceptions that learners may form in relation to performing operations.
- Activity 7 involved thinking about the "verbs of doing mathematics".
- Activity 10 involved mathematical concepts relating to multiplication, division and the solution of equations.
- Activity 13 involved mathematical concepts relating to number patterns.
- Activity 14 involved mathematical concepts relating to problem solving in a context that involved the geometric shape of a square, measurement and manipulation of the shape.
- Activity 15 involved mathematical concepts relating to ratio and rate in a problem solving context.

- Activity 16 involved mathematical concepts relating to interpretation of graphs representing data (horizontal bar graphs in particular).

- Activity 18 involved mathematical concepts relating to number patterns, some of which had geometric as well as numeric representations.