

Unit 3: Operations

Mathematics for Primary School Teachers

Work through this course to learn the mathematics you need to know for primary school teaching!

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Citation

Sapire, I. (2010). Mathematics for Primary School Teachers. **Saide** and the Wits School of Education, University of the Witwatersrand, Johannesburg.
ISBN: 978-0-9869837-5-7

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Unit 3: Operations

Introduction

In this unit, the four operations – addition, subtraction, multiplication and division – are discussed. Each operation is first introduced as a concept, after which different algorithms that can be used to perform the operations with ever-increasing efficiency, are then given and explained.

You are encouraged to teach your learners following a similar process, concepts first and then algorithms. But remember that ultimately, especially by the time the learners reach the senior phase, they should know and understand how the vertical algorithm works by employing place value simply and efficiently.

Upon completion of this unit you will be able to:



Outcomes

- *Explain and use* the algorithms for addition, subtraction, multiplication and division.
- *Demonstrate and illustrate* the use of various apparatus for conceptual development of the algorithms for addition, subtraction, multiplication and division.
- *Define and identify* multiples and factors of numbers.
- *Explain and use* the divisibility rules for 2, 3, 4, 5, 6, 8 and 9.
- *Discuss* the role of problem-solving in the teaching of operations in the primary school.
- *Apply* the correct order of operations to a string of numbers where multiple operations are present.

In the two sections that follow, we will look at the four operations. These four operations relate to a large part of the mathematical knowledge and skills that learners gain in the primary school. We will look at what is involved when each operation is performed, as well as a variety of methods (known as algorithms) for performing the operations.

All of the operations are known as **binary** operations because they are performed on two numbers at a time. If we need to operate on a string of numbers, we actually break the string up into pairs in order to do so.



Reflection

Think of solving the following sum:

$$2 + 17 + 28 + 73 =$$

How would you go about it?

Before we can introduce our learners to the operations, we need to be sure that they have a good concept of number, because the operations all work on numbers. Imagine trying to add 5 to 7, if you are still a little uncertain about exactly how much these symbols represent.

Whole number concept is usually well established in the foundation phase; teachers in higher phases need to ensure that bigger numbers, fractions and decimals are all soundly taught. This will help to eliminate the struggle that many learners have with operating on such numbers. Remember that we can use Piaget's test for conservation of number to check whether a sound number concept has been achieved (refer to Unit 2: Numeration).

When we operate on numbers, we get new numbers, depending on the operation performed. In order to get the correct new number, we need to have the correct understanding of the operation – this is what we first teach our learners, so that they will know what to do when called on to add, subtract, multiply or divide. We must teach them the terms, concepts, symbols and methods which are involved in each of the operations.

Background and algorithms for addition and subtraction

We can now move on to looking at addition and subtraction, and the algorithms (methods) for performing these operations.

There is very clear terminology which we should use if we wish to speak about the numbers involved when we add and subtract. You need to study this terminology so that you know it and use it. Remember that if we do not use correct mathematical terminology, we cannot expect our learners to do so!

All of the terms are given below:

$$\overset{\curvearrowright}{6} + \overset{\curvearrowright}{3} = \overset{\curvearrowright}{9}$$

Addends Sum

The concept of addition involves putting together certain amounts, to find out how much we have altogether. This amount is called the sum. You should try not to use the word "sum" incorrectly, to avoid unnecessary confusion for the learners.



Reflection

Where do we often use the word "sum" incorrectly?

Do you think that this misuse could be problematic for learners and if so in what way?

Subtraction

$$\begin{array}{ccccccc}
 \curvearrowright & 9 & - & 3 & \curvearrowleft & = & 6 & \curvearrowleft \\
 \text{Minuend} & & & \text{Subtrahend} & & & \text{Difference} &
 \end{array}$$

The concept of subtraction involves taking a given amount away from another given amount, to find out the difference between the two amounts.

You should notice that subtraction "undoes" what addition "does". Because of this relationship between the two operations, they are known as **inverse** operations. Give your own example that clarifies the meaning of addition and subtraction as inverse operations.

When we introduce any operation we should begin by using numbers in a real context – we tell stories that lead to the addition or subtraction of numbers. This makes it clear to the learners what they need to do, and also lays a foundation for their own problem-solving later, when they will have to read and interpret word problems.



Reflection

Give one example of an addition story and one for subtraction.

Another thing that we must do is show all the working that we do using concrete apparatus. We can use counters such as sticks or buttons, we could use paper pictures and Prestick on the board, or we could use felt boards and pictures, or other such apparatus depending on what is available.



Reflection

Illustrate and explain one addition and one subtraction question done using concrete apparatus.

Initial addition strategies are defined in terms of different ways of counting.

The first thing that learners do is called **counting all**. Here the learner groups together the two amounts he has to add and then counts how many items he has altogether, starting his counting from one.

The next strategy used is called **counting on**. Here the learner starts counting on from one of the addends (usually the first). For example, if he is required to add 5 and 3, he would start with 5 and count on 6, 7, up to 8. This represents some progress, as his counting does not have to go back all the way to one.

As the learner becomes more familiar with the adding process, he then moves on to **accelerated counting on**. Here he counts on in jumps, of 2 or 3 or 5 or 10 (etc.) depending on the sizes of the numbers involved in the question.



Activity

Activity 3.1

Give your own examples of:

1. Counting all
2. Counting on
3. Accelerated counting on.

Subtraction can also be worked out first by taking away in ones, and then by taking away in twos or bigger jumps, very similar to these initial adding strategies.

We move the learners on to recording their working using symbolic notation. Arrow diagrams can be useful in the recording of early counting, because arrow diagrams can show incomplete working. To establish good habits of recording mathematical working correctly, when learners use arrow diagrams it is good to get them to conclude their working with a correct mathematical sentence.

For example, a learner could write:

$8 + 5 \rightarrow 8 + 2 \rightarrow 10 + 3 \rightarrow 13$, and so $8 + 5 = 13$.

**Activity****Activity 3.2****Apparatus that can be used in the teaching of addition and subtraction**

For each apparatus mentioned give one practical way in which that apparatus could be used in the teaching of addition and subtraction.

1. Abacus – learners can manipulate the beads and observe the workings of the base 10 numeration system.
2. Dienes' blocks – these can be used for simple addition and subtraction of small numbers as well as for addition and subtraction of up to three digit numbers.
3. Unifix cubes – these are plastic blocks that can be stuck together and taken apart as desired. These are useful to use in introductory exercises, if available.
4. Hundred squares – a ten by ten square with the numbers from 1 to 100 laid out in rows. Can be used for counting on as well as for demonstrations of bigger number addition and subtraction, using accelerated counting on or taking away.
5. Number lines – single hops along a number line for early addition and subtraction examples, as well as bigger hops can be illustrated on a number line. The use of number lines is very good in consolidating number concept.

Drill

There has been some debate about whether or not conscious drilling of basic number facts is needed. We believe that learners who do not have a good grasp of all the basic number facts will be disadvantaged.

Addition and subtraction of all the single digit numbers, which can be extended to addition and subtraction of bigger numbers, is what we need to focus on. Drill sessions can be made into fun experiences for the children involving games, activities or competitions in groups or for the whole class. Drill must not be done in such a way that it puts the learners off learning, but rather in a way which excites them and assists them to learn and remember the essential number facts which they need to have at their fingertips.



Reflection

Brainstorm with some colleagues and record some ideas for interesting ways of drilling basic addition and subtraction facts.

Try your ideas out in the class!

There are certain laws of operations that you need to know about. Learners need not know the formal names of these laws, but they will be aware of them from quite an early stage. You need to know the names and functioning of these laws.

Remember that we have said that addition and subtraction are inverse operations. But addition and subtraction do not behave in exactly the same way.

Look at the following:

Addition is commutative: this means that we can add a pair of numbers in any order and still get the same answer.

For example, $5 + 9 = 9 + 5$, or in general we say that $a + b = b + a$.



Reflection

Give an example that demonstrates the commutativity of addition.

Subtraction in **NOT commutative**.



Reflection

Give an example that illustrates that subtraction is not commutative.

Addition is associative: this means that when we add three or more numbers together, we can pair them in any order we choose, without changing the final answer.

We write this in general as: $a + (b + c) = (a + b) + c$.



Reflection

Give a worked example that demonstrates the associativity of addition.

Subtraction is **NOT associative**.



Reflection

Give an example that illustrates that subtraction is not associative.

Both addition and subtraction have an **identity element**.

Addition has the identity element on the right and on the left, while subtraction only has a right identity element.

The identity element for addition and subtraction is zero, since:

$$0 + a = a + 0 = a, \text{ and } a - 0 = a.$$



Reflection

How would you explain what operating under the identity element results in?

Addition and subtraction algorithms

As we have mentioned before, algorithms are the methods of calculation of the operations.

We should encourage learners to experiment with different algorithms and we should therefore be able to do several different algorithms ourselves. Remember to be flexible and to accept any correct algorithm.

At all times insist on logical, meaningful work that you are able to interpret. Watch out for the use of correct mathematical sentences, and correct use of symbols.

Here are some different algorithms. Try them out yourselves – don't just read through them!

Vertical algorithm

| | | | |
|--|--|---|--|
| $\begin{array}{r} 728 \\ 345 + \\ \hline 1073 \end{array}$ | $\begin{array}{r} 139 \\ 128 - \\ \hline 11 \end{array}$ | $\begin{array}{r} 434 \\ 189 - \\ \hline 245 \end{array}$ | $\begin{array}{r} 312 \\ 268 - \\ \hline 44 \end{array}$ |
| Example 1: Regrouping | Example 2: No impasse | Example 3: Decomposition | Example 4: Equal additions |

Regrouping could be used in the first example because if you add 8 and 5 you get 13. This is actually one ten and 3 units. The units can be recorded in the units column, but the ten needs to be grouped and carried to the tens column, where it can be added to the other tens. Regrouping is done in any of the places, using base ten grouping of consecutive place values.

No impasse is how we describe the subtraction demonstrated in the second example. The subtraction can be done in columns without any decomposition since there are enough tens and units in the minuend to subtract the subtrahend without complication.

There is an **impasse** in examples 3 and 4.

Decomposition is one way in which one can overcome the impasse. This involved breaking down the tens in the minuend to add to the units of the minuend, so that the number of units in the minuend is greater than the number of units in the subtrahend. Decomposition is done in any of the places, using base ten grouping of consecutive place values.

Equal additions is another way of overcoming an impasse. This is done by adding ten units to the minuend while at the same time adding one ten to the subtrahend. This is a form of compensation, carried out in the vertical algorithm. Equal additions is done in any of the places, using base ten grouping of consecutive place values.

Horizontal algorithm

$$23 + 89 \rightarrow 23 + 9 \rightarrow 32 + 70 \rightarrow 102 + 10 \rightarrow 112$$

$$\text{i.e. } 23 + 89 = 112$$

$$348 + 741 \rightarrow 300 + 700 \rightarrow 1\,000 + 40 + 40 \rightarrow 1\,080 + 8 + 1 \rightarrow 1\,089$$

$$\text{i.e. } 348 + 741 = 1\,089$$



$$37 - 24 \rightarrow 7 - 4 \rightarrow 30 - 20 + 3 \rightarrow 10 + 3 \rightarrow 13$$

i.e. $37 - 24 = 13$

$$102 - 68 \rightarrow 100 - 60 \rightarrow 40 - 8 \rightarrow 32 + 2 \rightarrow 34$$

i.e. $102 - 68 = 34$

Compensation

$$18 + 26 = (18 + 2) + (26 - 2) = 20 + 24 = 44$$

$$1387 + 359 = (1\ 387 + 13) + (359 - 13) = 1\ 400 + 346 = 1\ 746$$

$$528 + 871 = (528 - 28) + (871 + 28) = 500 + 899 = 1\ 399$$



Reflection

What is a general rule that you could state for compensation under addition?

$$56 - 47 = (56 + 3) - (47 + 3) = 59 - 50 = 9$$

$$476 - 379 = (467 + 21) - (379 + 21) = 488 - 400 = 88$$

$$612 - 425 = (612 - 25) - (425 - 25) = 587 - 400 = 187$$



Activity

Activity 3.3

Now try some of your own algorithms.

You should be able to make up questions for yourself!

Write as many different examples as you can.

Background and algorithms for multiplication and division

In this section, multiplication and division are discussed. They are also binary operations (they act on two numbers at a time). Learners first need to establish the concept of the operation and then learn how to work with expressions involving these operations.



Multiplication

Multiplication is repeated addition of the same addend. We could introduce this idea of repeated addition by sketching or laying out the following using counters:

$$\begin{array}{ccccccccc} \circ \circ \circ & & \circ \circ \circ & & \circ \circ \circ & & \circ \circ \circ & & \circ \circ \circ \\ 3 & + & 3 & + & 3 & + & 3 & + & 3 & = & 15 \\ \text{Or five lots of three,} & & \text{or five threes,} & & & & & & & & 5 \times 3 = 15 \end{array}$$

By the time a learner knows that $5 \times 3 = 15$, they are not actually performing the operation, they are recalling this basic fact of multiplication from memory.

As with addition and subtraction, when we introduce multiplication to our learners, we should use number stories that will lead to simple multiples. Make up stories yourself and call on the learners to make up some of their own too. If you allow them to make up their own questions from this early stage, they will develop their independence and ability to think creatively about mathematical situations.



Reflection

Make up a story that would lead to a number sentence involving multiplication.

You need to know and use the terminology of multiplication that is recorded below. Study it carefully.

$$\begin{array}{ccccccc} \curvearrowright & 5 & \times & 3 & \curvearrowleft & = & 15 & \curvearrowleft \\ \text{Multiplier} & & & \text{Multiplicand} & & & \text{Product} \end{array}$$

You should use the correct language when you talk about multiplication. It is not correct to say "times this number by 7". The correct language is "multiply this number by 7". We do however speak about "7 times 5" which means "multiply 5 by 7".

Learners need a lot of drill in their multiplication tables. You as a teacher should certainly know all of the tables up to 9×9 . This would cover all the basic facts of multiplication

(all of the digit multiples from 1×1 to 9×9). Clearly, drilling of the basic facts is done after the operation concept has been thoroughly established. Remember also that the way in which drilling is done should be made exciting and interesting for the learners by using games and other classroom activities.

Like addition, multiplication adheres to certain of the laws of operations. You need to know these laws and what they imply.

Multiplication is commutative: This means that we can multiply a pair of numbers in either order, without changing the product.

For example, $7 \times 9 = 9 \times 7 = 63$.

You should be able to think of many such examples.

In general we write this as $a \times b = b \times a$.



Reflection

Can you think of an example that contradicts the statement that multiplication is commutative? If so, write it down.

Multiplication is associative: This means that if we have to multiply a string of three or more numbers, we can do so by pairing them in any order that we choose. In general we express this by writing $a \times (b \times c) = (a \times b) \times c$.



Reflection

Test the associativity of multiplication by calculating the following by pairing in different ways: $17 \times 50 \times 2 =$

Was there an order that was easier for you to do? If so, which one and why was it easier?

One of the strategies that we can teach our learners is to look out for "easier" ordering in questions involving multiplication of more than one number.

Multiplication has an **identity element:** This is the number which, when we multiply by, it has no effect on the multiplicand. The identity element of multiplication is the number 1, since, for example, $1 \times 8 = 8 \times 1 = 8$.

How would you express this in a general way?

Multiplication by zero is defined as follows: $0 \times a = a \times 0 = 0$

Multiplication is distributive over addition and subtraction

In general, we write this as follows:

over addition:

$$a \times (b + c) = (a \times b) + (a \times c) \text{ or } (b + c) \times a = (b \times a) + (c \times a)$$

over subtraction:

$$a \times (b - c) = (a \times b) - (a \times c) \text{ or } (b - c) \times a = (b \times a) - (c \times a)$$



Reflection

Try out the following example to check for the distributivity of multiplication:

$$13 \times (15 + 5) = 13 \times 15 + 13 \times 5 =$$

or

$$13 \times (15 + 5) = 13 \times 20 =$$

You should get the same answer in both cases.

Apparatus

Use apparatus such as Dienes' blocks and number lines.

Think about the use of calculators. When would you allow learners to use calculators in the class?

Algorithms

As with addition and subtraction, you need to be flexible and to encourage experimentation while insisting on correct, logical work at all times.

Learners can be encouraged to use horizontal and vertical algorithms as well as compensations. You must be able to use a variety of algorithms yourselves.



**Activity****Activity 3.4**

1. 32×15 (use a vertical algorithm)
2. 17×3 (use a horizontal algorithm)
3. Use compensations to calculate:
 - a) $16 \times 8 =$
 - b) $11 \times 21 =$
 - c) $14 \times 19 =$
4. What is wrong with saying "when we multiply, the number gets bigger"?

Division

There are two ways of conceptualising division. They are known as **grouping** and **sharing**. The two examples below will clarify the difference between the two conceptualisations.

**Example****Sharing division**

If I share 65 sweets among 7 children, how many sweets will each child get? The answer is $65 \div 7$ sweets. The way in which we do it is we share out the sweets, until they are all given out, and then we find out how many sweets each child got. This is known as sharing division. Many division word problems are phrased in this way.



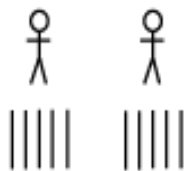
Example

Grouping division

I have 90 pieces of fudge. I want to sell them at the cake sale in little bags with 6 pieces of fudge per bag. How many bags can I make? The answer is $90 \div 6$ bags. But, to work it out, I put six pieces in each bag, until all the fudge is used up and then I count how many bags I was able to make. This is known as grouping division. This division strategy is often neglected, though it does occur in real situations.

Number questions out of a real context could be solved in a grouping OR a sharing way, because whether we think of 10 divided by 2 in a grouping or a sharing way, the answer is the same, only the strategy is different.

Look at the illustrations below and label them as "grouping" or "sharing" solutions to 10 divided by 2:



Each child gets
five sweets



There are 5 packets
with two sweets in each
packet

We must be sure to explain both division strategies to our learners.

The terminology for a division number sentence is recorded below. Be sure you know and use it.

$$\begin{array}{c}
 \curvearrowright \\
 15 \\
 \text{Dividend}
 \end{array}
 \div
 \begin{array}{c}
 3 \\
 \text{Divisor}
 \end{array}
 =
 \begin{array}{c}
 5 \\
 \text{Quotient}
 \end{array}$$

Division by zero and division of zero sometimes confuses learners. We define the following:

$$0 \div a = 0 \text{ (division of 0 leads to a quotient of 0)}$$

BUT $a \div 0$ is undefined (we cannot divide a number by zero)

Laws of operations and division

Division does **not** adhere to the commutative and associative laws, and it is not distributive over addition and subtraction. Division has a right identity element, since $16 \div 1 = 16$ (but $1 \div 16 \neq 16$, and so on the left, the identity element does not work).

Division and multiplication are **inverse operations**. This means that division "undoes" what multiplication "does".



Reflection

Give an example that illustrates multiplication and division as inverse operations.

Apparatus

Bottle tops and egg boxes, and pegs and boards, are the type of apparatus that can be used for early division exercises.

The children can also share items amongst themselves (in set groups).

For later questions on bigger numbers, Dienes' blocks can be used.

Algorithms

Once again a variety of algorithms should be encouraged and explained.

The horizontal algorithm is essentially a grouping strategy, while the old "long division" algorithm is a sharing strategy. Both are fairly long-winded and one needs to think of their application to reality.



Reflection

Many people have argued against pen and paper long division.
What would you say is its place in a mathematics education?



Activity

Activity 3.5

Try out some examples of your own such as the following which have been done for you:

1. Use a horizontal algorithm to solve the following

$$625 \div 18 =$$

$$625 - 180 \rightarrow 445 - 360 \rightarrow 85 - 36 \rightarrow 49$$

$$49 - 36 \rightarrow 13$$

$$10 + 20 \rightarrow 30 + 2 + 2 \rightarrow 34$$

$$625 \div 18 = 34$$

$$\left(\begin{array}{l} 18 \times 10 = 180 \\ 18 \times 20 = 360 \\ 18 \times 2 = 36 \end{array} \right)$$

2. Use a vertical algorithm to solve the following

$$583 \div 13 =$$

$$\begin{array}{r} 44 \\ 13 \overline{)578} \\ \underline{52} \\ 58 \\ \underline{52} \\ 6 \end{array}$$

$$583 \div 13 = 44 \text{ remainder } 6$$

3. Use compensations to solve the following

$$0,4 \div 0,2 = (0,4 \times 10) \div (0,2 \times 10) = 4 \div 2 = 2$$

$$150 \div 7,5 = (150 \times 2) \div (7,5 \times 2) = 300 \div 15 = 20$$

Multiples, factors and primes: divisibility rules

Multiples and factors are terms which we encounter in multiplication and division. Prime numbers are special numbers defined in terms of their factors.

These terms need to be understood so that they can be used, particularly in work on fractions. In this section, these and some other important related terms will be discussed.

Multiples



Learners use the idea of multiples when they do repeated addition, or drill their multiplication tables. One would expect this to be a commonly known term, and yet research has shown that it is not well understood at all. This may be because the term is not used sufficiently, and so you as educators need to be sure to use and explain the term **multiple** to your learners.



Example

Examples of multiples:

2, 4, 6, 8, 10 are multiples of 2

7, 14, 21, 28 are multiples of 7

The 60 minutes in an hour are broken up into multiples of 5 on a clock face that has standard markings.



Reflection

List the first 8 multiples of 9.

Common multiples

All numbers have an infinite number of multiples, while some pairs of numbers **share** certain multiples. To find common multiples, we simply write out some of the multiples of the given numbers and then look for those which are common to both.



Reflection

List the multiples of 2.

List the multiples of 3.

Numbers that are multiples of both 2 and 3 are called **common multiples** of 2 and 3.

**Reflection**

Which is the smallest of all the common multiples that you have circled?

This is called the **lowest common multiple**, or **LCM**, of 2 and 3.

**Activity****Activity 3.6**

Find the LCM of:

1. 2 and 5
2. 3 and 5
3. 2, 3 and 4
4. 2, 4 and 5.

Factors

A factor is a number that can divide completely into another number without leaving any remainder. We can find the factors for given numbers.

**Example**

The factors of 12 are 1, 2, 3, 4, 6 and 12.

The factors of 25 are 1, 5, and 25.

**Activity****Activity 3.7**

1. List the factors of 8.
2. List the factors of 16.
3. Now that you have listed the factors of 8 and of 16, can you identify the common factors of 8 and 16? List them.

**Reflection**

Which of the common factors you listed in the activity above is the biggest one?

This is the most important of the common factors, and it is called the **highest common factor** or **HCF**.

**Activity****Activity 3.8**

Find the HCF of:

1. 12 and 18
2. 15 and 30
3. 15, 20 and 35.

Prime numbers

Prime numbers are a special group of numbers which have only two **particular** factors. (Not all numbers have many factors like those in the examples chosen above). The two factors which prime numbers have are 1 and the number itself. For example: the factors of 7 are 7 and 1 and the factors of 23 are 23 and 1. These numbers are all examples of **prime** numbers.



Reflection

What are the factors of 31?

List a few other prime numbers that you can think of.

A **composite number** is any number that has at least one factor other than one and itself. Any number that is not a prime number is a composite number.



Reflection

What are the factors of 4?

What are the factors of 26?

Both 4 and 26 are examples of composite numbers.

The number 1 is **neither** prime, nor composite, since it has only one factor.

The **smallest prime** number, and only **even prime** number, is 2.

Prime factors

These are the factors of a number which are prime numbers in their own right. They are sometimes useful, since we can write a number entirely as a product of its prime factors.



Reflection

List the factors of 12

Which of these are prime numbers? List them. These are prime factors of 12.

**Activity****Activity 3.9**

1. What are the prime factors of 18?
2. Write the following numbers as a product of prime factors:
 - a. 8
 - b. 36
 - c. 51
 - d. 53
3. Twin primes are primes with a difference of 2, such as 3 and 5, and 11 and 13. How many pairs of twin primes are there in the first 100 natural numbers?

Divisibility rules

The section on divisibility rules also falls within our study of the operations since it relates to division of certain numbers by other numbers. Divisibility rules are rules (some are simple, some more complex) whereby we are able to check whether certain numbers are divisible by other numbers (without leaving a remainder) **WITHOUT** actually dividing. Divisibility rules therefore enable us to check quickly whether a number is a factor of another number.

You will all know the rule for divisibility by 2, even if you are unaware of it. The rule is to **examine the last digit of the given number**: if it is even, the given number will be divisible by 2, if it is odd, the number will not be divisible by 2.

For example, 49 553 is not divisible by 2, but 49 554 is divisible by 2 ... you should not actually have to divide to know this.

Think of any number, and you will immediately be able to say whether it is divisible by two or not. This is because you know the divisibility rule for two.

Other divisibility rules that you should know are those for 5 and 10. They also have to do with examining the last digit(s) of the given number. You should be able to apply these rules with hardly any effort at all, because they are simple and familiar to you.

**Reflection**

Think of the numbers 346, 380, 865, 1000 and 991.

List those that are divisible by 5.

List those that are divisible by 10.

This should not have required any calculation on your behalf, and should not have taken you very long to do.

We have discussed these commonly known divisibility rules to make the nature of divisibility rules clear to you. They make checking for divisibility by a certain number into a quick, easy mental checking process, and because of this they are useful.

There are rules for divisibility by 3, 4, 6, 8 and 9 that you must also know.

The rules for divisibility by 4 and 8 are related to the rule for divisibility by 2, because 2, 4, and 8 are all powers of 2.

To check for **divisibility by 4** examine the last two digits of the given number. If those two numbers (on their own, ignoring all the other digits in the given number) form a number that is divisible by 4 then the whole original number is divisible by 4.

**Reflection**

In what way is this similar to the divisibility rule for 2?

To check for **divisibility by 8** examine the last 3 digits of the given number (on their own, ignoring all the other digits in the given number). If these three digits form a number that is divisible by 8 then the whole original number is divisible by 8.

**Activity****Activity 3.10**

Check the following numbers for divisibility by 4 and by 8:

1. 386
2. 1000
3. 572 450
4. 1 259 080

The **divisibility rules for 3 and 9** are a little different, and slightly more complicated, but learners enjoy them, and once you have grasped them, it is not difficult to apply them fairly quickly. It is certainly quicker than doing an actual division of the bigger numbers.

To check for divisibility by 3 and 9, we **use the face values of the digits**.

If, when the face values of all the digits in the number are added, the sum obtained is divisible by 3, then the whole number is divisible by 3.

For example, 304 233 is divisible by 3, since $3 + 0 + 4 + 2 + 3 + 3 = 15$, which is a multiple of 3, and hence is divisible by 3. Check this for yourselves!

If, when the face values of all the digits in the number are added, the sum obtained is divisible by 9, then the whole number is divisible by 9.

When you are adding the face values of the digits, you do not have to keep a running total, as you reach multiples of 3 (or 9 depending on what you are checking for) you can cast out or drop these multiples, and so keep the addition very simple.

**Activity****Activity 3.11**

1. Check the following numbers for divisibility by 3 and 9:
 - a. 111 080 234
 - b. 876 957 642
 - c. 111 111 111
2. Make up other numbers for yourselves, write them down, and check whether or not they are divisible by 3 and 9.

To check for **divisibility by 6** we have to check for divisibility by 2 and 3. If the given number is divisible by both 2 and 3, it will also be divisible by 6. This might be a little

time consuming, but it still only involves simple mental checking, which does not actually take too long. For example, an easy number to think of is 66 – it is divisible by 2 and by 3, and as you should know, it is also divisible by 6.

The rule is useful for bigger numbers, such as those in the problems below:



Activity

Activity 3.12

Which of the following are divisible by 6?

1. 2 937 810
2. 607 001
3. 345 102

This rule for divisibility by 6 works because 2 and 3 are relatively co-prime. This means that the numbers 2 and 3 have no common factors apart from the number 1. The rule is true for **all pairs** of relatively co-prime factors of other given numbers.

For example, using this extension of the rule, you can also check for divisibility by 24, if you check for divisibility by 8 and by 3 (since 8 and 3 have no common factors apart from the number 1).



Activity

Activity 3.13

1. Is 203 016 divisible by 24?
2. How would you check for divisibility by:
 - a. 45?
 - b. 42?
 - c. 18?

The **divisibility rule for 7** is fairly long and tedious. There is one and it does work, but it's not a "time saver" by any means! Here it is for your interest. You do not need to learn this one for your exam.

Step 1: Remove the last digit to obtain a new number

Step 2: Take twice the last digit away from this number

Step 3: If the result of step 3 is divisible by 7, the original number is divisible by 7



Note that if, after steps 1 and 2, the divisibility or otherwise of the number by 7 is not obvious, repeat steps 1 and 2 on the result of step 2.



Activity

Activity 3.14

Test the following numbers for divisibility by 7:

1. 45 607
2. 62 692
3. 31 934
4. 2 448 810

Learners especially enjoy the rule for **divisibility by 9**. It is easy and almost "magical" and puts some fun into division. Why not introduce some of these rules to your learners and see how they like them!

Problem-solving, word problems and order of operations

Problem-solving

Solving problems is one of the best exercises you can give to your brain. Mathematics is a subject that lends itself to problem-solving activities. We can exercise our own brains (and those of our learners) if we apply our minds to mathematical problems. "Word problems" (often spoken about in maths) are branded by some people as impossible and not worthy of spending time on. People with this attitude are cutting back on the potential impact of mathematics on the development of their learners.

Problem-solving is not an activity which should be reserved for an elite few – we can develop our skills of problem-solving through following guidelines for the structuring of their solutions and through **perseverance**. We must not expect to solve all problems in a few minutes – this would be unrealistic. Not all problems are so simple! Some do require deep thought and careful consideration in order to be solved. This is where problem-solving trains us for real life, and where our mathematics training can be seen as equipping us for everyday situations and some of the problems we are confronted with in life.

You need to approach a problem systematically. Consider the following steps which could guide you towards successful problem-solving.

Step 1:

Read the problem carefully and ensure that you understand what the problem is about. Restating the problem in your own words is a good exercise, which will make it clear to

you whether or not you have understood the meaning of the problem. It is often a good idea to try and sketch a diagram that assists you to illustrate what is required by the problem.

Step 2:

Once you have understood what the problem is asking, you have to think of your strategy for solving the problem. Think about what operations you may have to use in the solution. Have you got all the information that you need in order to solve the problem? Have you solved other similar problems which can guide your solution to the current problem? And, can the problem be broken up into smaller parts if it seems too big to solve all at once?

Step 3:

This step should not present you with large problems if step 2 has prepared you adequately to solve the problem. Here you go about implementing your problem-solving strategy to get to the actual solution to the problem. It is important that you realise the difference between devising a strategy to solve a problem and the actual solution to the problem. Both are important activities. It will become clear to you if you need to change your strategy or find a new one, or if your original strategy was adequate.

Step 4:

Once you have solved the problem, a final "logic check" of your solution is never a waste of time. Careless errors can slip into your working (though your strategy may be correct) and lead you to an answer which is not correct. Re-read your work just to be sure that it makes sense and presents a valid, satisfactory solution to the problem. This step of verification may seem like a waste of time, but will often prove its usefulness when on verification, you make small changes and improvements to your answers.

**Reflection**

How would you summarise each of steps 1 to 4 above in one sentence or phrase?

In this course you will have to demonstrate your problem-solving ability. In your classroom you will have to develop the problem-solving ability of your learners. To do so, you will have to ensure that you set them sufficient challenging problems and insist on the learners themselves solving these problems. **DO NOT DO THE WORK FOR THEM.**



You will also have to ensure that your class is divided into functional groups, where meaningful interaction occurs. Problem-solving can (and should) be an interactive activity: do not leave the learners on their own to solve all the problems you set them; let them work in groups (pairs, fours, etc., depending on the nature of the problem and the size of the class).



Reflection

Why do you think problem-solving is such a good activity in a maths classroom?

Why do you think group work is valuable in a problem-solving context?

There are different types of problems that we can set. We should try to include a range of problems, rather than set problems that are essentially the same all of the time.

Problems will usually have some relevance to real life situations though some can be abstract and call for thought on a more abstract level.

Many problems set in maths have a single solution. As a real life model, this is not adequate, since many real life problems have more than one solution depending on varying circumstances. We need to try and include some multiple solution problems in our range of problems set.

The same applies to ambiguity in problems: not all problems should be simple and straightforward, as this does not equip our learners for the ambiguities that arise in real life.

Problems need to be graded and we need to include problems from the most simple to those which are more complex and difficult, if we are to give our learners experience in solving the range of problems which they might encounter in real life.

As we said earlier, you need to be confident about problem-solving yourself.

Here are some problems for you to try. The very best way to improve your own problem-solving skill is to put it to the test. Remember that advice from friends on the solutions can be useful, but don't rely too much on others – exercise your own brain as much as possible! Look back at the guidelines for problem-solving strategies at this stage – they might be helpful.

The problems below are on mixed operations and mixed levels. You should work through all of these problems, the solutions to which you will of course receive at a later stage.

**Activity****Activity 3.15**

1. There were 563 learners in the school hall for assembly. The juniors (297) went back to class early. How many seniors were left in the hall?
2. There were 20 shoes in a cupboard. How many pairs of shoes were there?
3. Mbongeni is mad about jigsaw puzzles. He has one puzzle with 500 pieces, one with 1 250 pieces and a real giant with 2 500 pieces. How many puzzle pieces are there in these three puzzles altogether?
4. Find the difference between the product of 3 and 31 and the product of 2 and 46.
5. You have 364 apples and you want to put them into 31 packets. How many apples would you put into each packet?
6. The distance from Johannesburg to Cape Town is approximately 1500km. If I have done 786 km of the journey, about how far do I still have to go?
7. Sphiwe and Rose had to fold serviettes for their older sister's wedding. Their little brother ran past with a jug of water and fell, spilling water on some of the 232 serviettes that they had folded. They quickly sorted them, but still had to throw away 87 serviettes. How many folded serviettes do they still have that they can use for the wedding celebration?
8. Franklin Primary School has 3 Grade 7 classes with 31 learners in each class. There are only 2 Grade 4 classes, each with 52 learners. Are there more Grade 4 or Grade 7 learners? How many more?
9. Thembi can type 60 words per minute. How many words can she type in 35 minutes?
10. **THIS ONE IS A CHALLENGE:** You are given 4 separate pieces of chain that are each 3 links in length. It costs R2,20 to open a link and R1,30 to close a link. All the links are closed. What is the cheapest possible way of making a single closed chain?

Order of operations

The order in which we perform the operations when we are operating on more than one pair of numbers, is determined by a few simple rules.



If you do not adhere to the rules, you might come up with incorrect answers, not because you have performed the wrong operation, but because you did so in the wrong order. It is therefore crucial that you know and are able to teach the correct order of operations.

The first (and general) rule for operating on a string of numbers which has 3 or more terms, is to **work from left to right**.

There are some instances where we go against the "left to right" rule, but these instances are set. We cannot do so in any way we choose.

The word BODMAS is often used to assist us to remember that certain operations take priority over other operations when they appear together in the same string. We interpret BODMAS in the following way:

B stands for brackets. Anything which is in brackets must be evaluated before anything else in the expression.



Example

$3 \times 5 + 2 = 15 + 2 = 17$ is different to

$3 \times (5 + 2) = 3 \times (10) = 30$, where brackets prioritise the addition of 5 and 2.

D and M stand for division and multiplication. These must be done BEFORE addition and subtraction, which are represented by the letters A and S.



Example

$3 \times 4 + 5 \times 6 = 12 + 5 \times 6 = 17 \times 6 = 102$ is NOT correct, while

$3 \times 4 + 5 \times 6 = 12 + 30 = 42$ is the correct order in which to perform this operation string.

The multiplication is done BEFORE the addition, against the general "left to right" rule. This is how it has to be done here.



Note

If division and multiplication appear together, you simply work from left to right. Division does not come before multiplication. Similarly, if addition and subtraction appear together, again, you simply work from left to right. Addition does not come before subtraction.



Example

Study the following examples carefully, watching the order in which the operations are performed:

1. $16 \times 8 \div 2 = 128 \div 2 = 64$

2. $50 + 6 - 15 + 1 - 3 - 15 = 56 - 15 + 1 - 3 - 15 = 41 + 1 - 3 - 15 = 42 - 3 - 15 = 39 - 15 = 24$

3. $6 \times 2 - 3 \times 1 = 12 - 3 = 9$

4. $7 \div 1 + 6 = 7 + 6 = 13$

5. $4 - 2 \times 2 = 4 - 4 = 0$

6. $5 + 3(2 + 2(4 + 3)) = 5 + 3(2 + 2(7)) = 5 + 3(2 + 14) = 5 + 3(16) = 5 + 48 = 53$

7. $2(14 - 3(7 - 6) - 5 + 10) = 2(14 - 3(1) - 5 + 10) = 2(14 - 3 - 5 + 10) = 2(11 - 5 + 10) = 2(6 + 10) = 2(16) = 32$

**Activity****Activity 3.16**

Now try the following examples yourself. Write down all of your working, so that when you check your work you are able to pick up where you made your mistakes. The numbers chosen for the examples are all small because the focus here is on the order of the operations, not the difficulty of the computation with the numbers involved.

1. $100 - 18 - 8 =$

2. $7 \div 50 \div 2 =$

3. $30 + 5 \times 15 =$

4. $16(2(4 \div 2)) =$

5. $6 \div 3 \times 4 \div 2 + 1 =$

6. $(12 + 5) \times 100 =$

7. $20 + 3 - 2 + 5 - 7 =$

8. $4 \times 7 \times 3 \div 2 + 1 =$

9. $4 + (20 - 7) + 12 =$

10. $45 \div 9(5 + 1) =$

Unit summary



In this unit you learned how to:

- *Explain and use* the algorithms for addition, subtraction, multiplication and division.
- *Demonstrate and illustrate* the use of various apparatus for conceptual development of the algorithms for addition, subtraction, multiplication and division.
- *Define and identify* multiples and factors of numbers.
- *Explain and use* the divisibility rules for 2, 3, 4, 5, 6, 8 and 9.
- *Discuss* the role of problem-solving in the teaching of operations in the primary school.
- *Apply* the correct order of operations to a string of numbers where multiple operations are present.

Assessment



Operations

1. Explain what is meant by the following statements which refer to the different numeracy skills. Remember to go into sufficient detail.
 - a. Learners should know, by heart, number facts such as number bonds up to 20, multiplication tables up to 10×10 , division facts, doubles and halves.
 - b. Learners should be able to calculate accurately and efficiently, both mentally and on paper, drawing on a range of calculation strategies.
 - c. Learners should be able to make sense of number problems including non-routine problems, and recognise the operations needed to solve them.
2. Suggest a creative activity that could be used to drill the multiplication tables.
3. Solve the following using compensations: (Show your working.)
 - a. $495 + 382$
 - b. $649 - 478$
 - c. 25×16
 - d. $328 + 874$
 - e. $295 \times 0,4$
4. Solve the following using horizontal algorithms:
 - a. 32×67
 - b. $254 + 14$
5. Solve the following using vertical algorithms:
 - a. $4\,967 + 3\,424$
 - b. $495 \div 15$ (use long division)
 - c. 314×26